Transformation of Conformal Coordinates of Type Lambert Conic from a Global Datum, (ITRS) to a Local Datum (regional, National)

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Key words: Global geodetic datum, ITRS, Local geodetic datum, Lambert conic projection, Curvilinear datum transformation.

SUMMARY

Local geodetic systems with appropriate choices of map projections have been developed in the past, in order to satisfy the surveying and mapping requirements of countries all over the earth. In most of these geodetic systems, the horizontal and vertical points are separated in location, monumentation, and measurement, and are referred to different datums. Local horizontal geodetic points have approximate or no heights, and benchmarks have very weak horizontal coordinates. In recent years, a growing trend toward the use of GPS observations and global mapping satellite systems have resulted in position based products in a world reference frame. There is therefore need in geodetic practice to transform coordinates referred to the local geodetic system to the global geodetic system and vice versa. Also a fundamental activity in land surveying is the integration of multiple sets of geodetic data, gathered in various ways, into a single consistent data set, that is into a common geodetic reference frame. Thus geodetic coordinate transformations find application in several instances, such as navigation, revision of older maps, cadastral surveying, GIS, industrial surveying, deformation studies, geo-exploration, etc.

This paper treats the problem of how to transform from global datum, for instance from the International Terrestrial Reference system (ITRS) to a local datum, for instance of type regional or national, for the practical case of the Lambert projection of the sphere or the ellipsoid-of-revolution to the cone. We design the two projection constants $n(\varphi_1, \varphi_2)$ and $m(\varphi_1)$ for the Universal Lambert Conic project of the ellipsoid-of-revolution. The task to transform from a global datum with respect to the ellipsoid-of-revolution $\mathrm{E}^2_{a,b}$ to local datum with respect to the alternative ellipsoid-of-revolution $\mathrm{E}^2_{a,b}$, without local ellipsoidal height, is solved by an extended numerical example.

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1. INTRODUCTION

The *Lambert projection of the sphere to the cone* is one of the most popular map projections. Here we generalize it to the ellipsoid-of-revolution and study in detail the equations which govern a datum transformation extend by form parameters. The inclusion of form parameters is needed when we switch from a global datum, for instance from the *International Terrestrial Reference System* (ITRS) to a local datum (regional, National) and vice versa.

Section 2 of this paper is devoted to generalize the classical Lambert conic project of the sphere to the ellipsoid-of-revolution and introduce the datum transformation given in the appendix as a curvilinear transformation characterized by seven parameters (three for translation, three for rotation, one or scale).

At first, we introduce by Definition 1.1 the Universal Lambert Conic Projection of the ellipsoid-of-revolution in local coordinates ($\overline{x}, \overline{y}$) or ($\overline{\alpha}, r$). The latitudes of two parallel circles are mapped equidistantly. They are referred as two projection constants of the first kind and of the second kind respectively, here called $n(\varphi_{1,}\varphi_2)$ and $m(\varphi_{1,})$.

Secondly, in order to transform Lambert conformal coordinates given in a global datum with respect to the ellipsoid-of-revolution $\boldsymbol{E}^2_{A,B}$ into Lambert conformal coordinates in a local datum with respect to the *ellipsoid-of-revolution* $\boldsymbol{E}^2_{a,b}$ we take advantage of the Taylor expansion of second order. Given ellipsoidal coordinates of a point by means of $\{\Lambda, \Phi, \mathrm{H}\}$, for instance by GPS, GLONASS or others, we design the corresponding matrices K and A which build up the 7 datum parameters and the 2 form parameters of transformation to a local datum. We collect the various results in section 2.1 – section 2.3.

The third section is based on a detailed example. We start from seven datum parameters, ellipsoidal parameters, and from two values of latitudes of types parallel circles in Tables 1a and 1b. Global and local sets of coordinates $\{\Lambda, \Phi, \mathrm{H}\}$, and (λ, φ) are represented by Table 2 and Table 3, while datasets of type Eastings and Northings of 14 points are given in Table 4. Table 5 contains the computed values of Eastings and Northings based on equation (11) and (12) without including the second order terms and their corresponding difference (dE, dN). Their differences are in the centimeter range and therefore cannot be neglected.

We took care of our global representation of ellipsoidal heights in terms of ortho normal functions (Grafarend and Engels 1992). The datum transformations including the form parameters are referred to contributions of Grafarend, Krumn and Okeke (1995), Grafarend and Okeke (1998), Grafarend and Syffus (1998), Leick and van Gelder (1975), Okeke (1997), and Soler (1976). Special attention is on our contribution, Grafarend and You (1998), regarding the Newton form of a geodesic in Maupertuis gauge on the sphere and on the ellipsoid-of-revolution.

2. EQUATION GOVERNING DATUM TRANSFORMATION EXTENDED BY FORM PARAMETERS OF THE LAMBERT CONIC PROJECTION (CONFORMAL)

Definition 1.1 (Universal Lambert Conic Projection, local coordinates of $E^2_{a,b}$ on □$^2_{a,b}$): A conformal transformation of ellipsoidal coordinates of type "surface normal" ellipsoidal longitude l and "surface normal" ellipsoidal latitude φ into Cartesian coordinates $\tilde{x}, \tilde{y}$ with respect to a local ellipsoid of revolution $E^2_{a,b}$ is called Universal Lambert Conic Projection if

(1) $$\tilde{x}, = r\cos\tilde{\alpha}, \quad \tilde{y} = r\sin\tilde{\alpha}$$

(2) $$\tilde{\alpha} = nl$$

(3) $$r = am[\tan(\frac{\pi}{4} - \frac{\varphi}{2})(\frac{1+e\sin\varphi}{1-e\sin\varphi})^{e/2}]^m =: r(\varphi)$$

subject to

(4) $$m = m(\varphi_1) := \frac{\cos\varphi_1}{n\sqrt{1-e^2\sin^2\varphi_1}}[\tan(\frac{\pi}{4} - \varphi_1)(\frac{1+e\sin\varphi^1}{1-e\sin\varphi^1})^{e/2}]^{-n}$$

(5) $$n = n(\varphi_1, \varphi_2) := \frac{\ln[\cos\varphi_2(1-e^2\sin^2\varphi_1)^{1/2}] - \ln[\cos\varphi_1(1-e^2\sin^2\varphi_2)^{1/2}]}{\ln[\tan(\frac{\pi}{4} - \frac{\varphi_1}{2})(\frac{1+e\sin\varphi_1}{1-e\sin\varphi_1})^{e/2}] - \ln[\tan(\frac{\pi}{4} - \frac{\varphi_2}{2})(\frac{1+e\sin\varphi_2}{1-e\sin\varphi_2})^{e/2}]}$$

holds. We have denoted by a the semi-major axis, by b the semi-minor axis, by $e:\sqrt{1-b^2/a^2}$ the relative eccentricity of $E^2_{a,b}$. (λ, φ) are the curvilinear coordinates which cover the elliptic cone □$^2_{a,b}$ which is developed onto $\mathbf{R}^{2^2}$ covered by its polar coordinates $(\tilde{\alpha}, r)$. $(\varphi_{1,}\varphi_2)$ are the latitudes of those parallel circles (coordinate lines $\varphi_{1,}$ =const and φ_2 =const) which are mapped equidistantly. $n(\varphi_{1,}\varphi_2)$ and $m(\varphi_{1,})$ are conveniently called projection constants of the first kind and of the second kind, respectively.

In order to transform Lambert conformal coordinates given in a global datum with respect to the ellipsoid of revolution $E^2_{A,B}$ into *Lambert conformal coordinates* in a local datum with respect to the ellipsoid of revolution $E^2_{a,b}$ we take advantage of the *Taylor expansion* of second order.

(6) $$\lambda = \Lambda + \delta\Lambda,\ \varphi = \Phi + \delta\Phi,\ a = A + \delta A,\ e = E + \delta E$$
so that

(7) $$\tilde{\alpha}(\lambda, e) = \tilde{\alpha}(\Lambda + \delta\Lambda, E + \delta E) = \tilde{\alpha}(\Lambda, E)\Lambda + \frac{d\overline{\alpha}}{d\Lambda}\delta\Lambda + \frac{d\overline{\alpha}}{dE}\delta E + O_{2\overline{\alpha}}$$

(8) $$r(\varphi, a, e) = r(\Phi + \delta\Phi; A + \delta A, E + \delta E) = r(\Phi; \mathrm{A}, \mathrm{E}) + r_\varphi \delta\Phi + r_a \delta A, + r_e \delta E + O_{2r}$$
or
(9) $$\tilde{x} = r\cos\tilde{\alpha} = r(\Phi + \delta\Phi; A + \delta A, E + \delta E)\cos\tilde{\alpha}(A + \delta A, E + \delta E)$$

(10) $$\tilde{y} = r\sin\tilde{\alpha} = r(\Phi + \delta\Phi; A + \delta A, E + \delta E)\sin\tilde{\alpha}(A + \delta A, E + \delta E)$$

(11) $$\begin{aligned}\tilde{x} = {} & r(\Phi; \mathrm{A}, \mathrm{E})\cos\tilde{\alpha} - r(\Phi; \mathrm{A}, \mathrm{E})n\sin\tilde{\alpha}\delta\Lambda + \\ & + r_\varphi(\Phi; \mathrm{A}, \mathrm{E})\cos\tilde{\alpha}\delta\Phi + r_a(\Phi; \mathrm{E})\cos\tilde{\alpha}\delta\mathrm{A} + \\ & + (-r(\Phi; \mathrm{A}, \mathrm{E})\sin\tilde{\alpha}\tilde{\alpha}_e(\mathrm{E}) + \cos\tilde{\alpha}r_e(\Phi; \mathrm{A}, \mathrm{E}))\delta\mathrm{E} + O_{2\bar{x}}\end{aligned}$$

(11i) $$\begin{aligned}O_{2\tilde{x}}(\Phi; \mathrm{A}, \mathrm{E}) = {} & \left(\frac{-r\sin\tilde{\alpha}\Lambda n_e}{A} + \frac{\cos\tilde{\alpha}r_e}{A}\right)\delta(A, \mathrm{E}) + \frac{\cos\tilde{\alpha}r_\varphi}{A}\delta(A, \Phi) + \frac{-rn\sin\tilde{\alpha}}{A}\delta(A, \Lambda) \\ & -(\cos\tilde{\alpha}nr\Lambda n_e + r\sin\tilde{\alpha}n_e + n\sin\tilde{\alpha}r_e)\delta(\Lambda, E) + n\sin\tilde{\alpha}r_\varphi)\delta(\Lambda, \Phi) - \frac{1}{2}n^2 r\cos\tilde{\alpha}\delta\Lambda^2.\end{aligned}$$

(12) $$\begin{aligned}\tilde{y} = {} & r(\Phi; \mathrm{A}, \mathrm{E})\sin\tilde{\alpha} + r(\Phi; \mathrm{A}, \mathrm{E})n\cos\tilde{\alpha}\delta\Lambda + r_\varphi(\Phi; \mathrm{A}, \mathrm{E})\sin\tilde{\alpha}\delta\Phi + \\ & r_a(\Phi; \mathrm{E})\sin\tilde{\alpha}\delta\mathrm{A} + (r(\Phi; \mathrm{A}, \mathrm{E})\cos\tilde{\alpha}\tilde{\alpha}_e(\mathrm{E}) + \sin\tilde{\alpha}r_e(\Phi; \mathrm{A}, \mathrm{E}))\delta\mathrm{E} + O_{2\tilde{y}}\end{aligned}$$

(12i) $$\begin{aligned}O_{2\tilde{y}}(\Phi; \mathrm{A}, \mathrm{E}) = {} & \left(\frac{-r\cos\tilde{\alpha}\Lambda n_e}{A} + \frac{\sin\tilde{\alpha}r_e}{A}\right)\delta(A, \mathrm{E}) + \frac{\sin\tilde{\alpha}r_\varphi}{A}\delta(A, \Phi) + \frac{-rn\cos\tilde{\alpha}}{A}\delta(A, \Lambda) \\ & -(\sin\tilde{\alpha}nr\Lambda n_e + r\cos\tilde{\alpha}n_e + n\cos\tilde{\alpha}r_e)\delta(\Lambda, E) + n\cos\tilde{\alpha}r_\varphi)\delta(\Lambda, \Phi) - \frac{1}{2}n^2 r\sin\tilde{\alpha}\delta\Lambda^2\end{aligned}$$
Where
$$\tilde{\alpha} = n(E)\Lambda;\ \tilde{\alpha}_e(\Lambda, E) := \frac{d\tilde{\alpha}}{de}(\Lambda, E) = \Lambda n_e;\ n_e(E) := \frac{dn}{de}(E)$$

These expressions up to second order O_{2r} or O_{2x}, O_{2y} include the first derivatives

(13) $$r_\varphi(\Phi; \mathrm{A}, \mathrm{E}) := \frac{dr}{d\varphi}(\Phi; \mathrm{A}, \mathrm{E})$$

(14) $$r_a(\Phi; \mathrm{E}) := \frac{dr}{da}(\Phi; \mathrm{E})$$

(15) $$r_e(\Phi; \mathrm{A}, \mathrm{E}) := \frac{dr}{de}(\Phi; \mathrm{A}, \mathrm{E})$$

as well as the longitude increments $\delta\Lambda$,latitude increments $\delta\Phi$, increment of semi-major axis δA and of eccentricity δE . The zero order terms includes the radial functions $r(\Phi; A, E)$ with respect to the global semi-major axis A and global eccentricity E. The projection constants $m(\varphi_1), n(\varphi_1, \varphi_2)$ are fixed under a datum transformation by means of $\varphi_1 = \Phi_{1,} \varphi_2 = \Phi_2$. In addition, second order terms containing second order variation of the constant terms $m(\varphi_1, E), n(\varphi_1, \varphi_2, E)$ with respect to the eccentricity E are neglected.

Theorem 1.2 (Datum transformation, Universal Lambert Conic Projections of $E^2_{a,b}$ towards $E^2_{A,B}$): Let the seven datum parameters of type translations, rotations, scale and two form parameters of type semi-major axis, relative eccentricity between a local and global reference system be given. Then polar ULC coordinates transform according to (16), (17) while Cartesian ULC coordinates transform according to (21), (22).

We shall be able to switch from the global ULC coordinates (polar or Cartesian), to the local ULC coordinates polar or Cartesian) or vice versa.

2.1 First Variation Of The Radial Coordinate $r(\varphi; a, e)$

The first variation of the radial coordinate is given by equation (13i)

(13i) $$r_\varphi = amn(r1r2)^{-1+n\frac{d(r1r2)}{d_\varphi}}$$

Where:

$$r1 := \tan(\frac{\pi}{4} - \frac{\varphi}{2})\,;\; r2 := (\frac{1+e\sin\varphi}{1-e\sin\varphi})^{e/2}\,;\; \frac{d(r1r2)}{d_\varphi} = \frac{-r2}{2} - \frac{r1^2 r2}{2} + \frac{e^2 r1r2\cos\varphi}{1-e^2\sin^2\varphi}$$

(14i) $$r_a = m[\tan(\frac{\pi}{4} - \frac{\varphi}{2})(\frac{1+e\sin\varphi}{1-e\sin\varphi})^{e/2}]^n$$

(15i) $$re = r(\frac{\frac{dc}{de}}{c} + \ln(r1r2)n_e + \frac{n\frac{dr2}{de}}{r2})$$

Subject to

$$\frac{dr^2}{de} = \frac{\ln(r2^{2/e})r2}{2} + \frac{er2\sin\varphi}{1-e^2\sin^2\varphi}$$

$$c = am;\; c2 := (1- e^2\sin^2 j\,)^{1/2}; c3 := \tan(\frac{p}{4} - \frac{j}{2});\; c4 := (\frac{1+ e\sin j_1}{1- e\sin j_1})^{e/2}$$

$$\frac{dc}{de} = c(-\frac{\frac{dc2}{de}}{c2} - n\frac{\frac{dc4}{de}}{c4} - \ln(c3c4)n_e - \frac{n_e}{n});\; \frac{dc2}{de} = \frac{e\sin^2\varphi_1}{\sqrt{(1-e^2\sin^2\varphi_1)}};\frac{dc4}{de} = \frac{\ln(c4^{2/e}c4)}{2} + \frac{ec4\sin\varphi}{1-e^2\sin^2\varphi_1}$$

(15ii) $$ne := \frac{dn}{de} = -\frac{nl\,\frac{dnu}{de} - nu\,\frac{dnl}{de}}{nl_2}$$

Subject to

$$nu = \ln[\cos\varphi_2(1-e^2\sin^2\varphi_1)^{1/2}] - [\ln[\cos\varphi_1(1-e^2\sin^2\varphi_2)^{1/2}]$$

$$nl = \ln[\tan(\frac{\pi}{4}-\frac{\varphi_1}{2})(\frac{1+e\sin\varphi_1}{1-e\sin\varphi_1})^{e/2}] - \ln[\tan(\frac{\pi}{4}-\frac{\varphi_2}{2})(\frac{1+e\sin\varphi_1}{1-e\sin\varphi_1})^{e/2}]$$

$$\frac{dnu}{de} = \frac{e\sin^2\varphi_1}{e^2\sin\varphi_1 - 1} - \frac{e\sin^2\varphi_2}{e^2\sin\varphi_2 - 1}$$

With the first derivatives of the radial polar coordinate $r(\varphi; a, e)$ given in section 2.1 we are able to initiate the 7-parameter datum transformation reviewed in the *Appendix*, namely by implementing the *curvilinear datum transformation* $\delta\Lambda$ (A17i), $\delta\Phi$ (A17ii), a procedure summarized in sections 2.2 and 2.3.

2.2 Datum Transformation of Universal Lambert Conic Projection (ULC) - Polar form

Let $\tilde{\alpha}$ be given as:

$$\tilde{\alpha} = n(E)\Lambda\frac{d\tilde{\alpha}}{d\Lambda}\delta\Lambda + \frac{d\tilde{\alpha}}{d\mathrm{E}}\delta\mathrm{E}.$$

Then

(16) $\tilde{\alpha} = n(E)\Lambda + n(E)n(E)(k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14}\alpha + a_{15}\beta + a_{16}\gamma + a_{17}s) + \Lambda n_e(E)\delta E.$

Also

(17) $$r = R(\Phi; \mathrm{A}, \mathrm{E}) + r_\varphi\delta\Phi + r_a\delta A + r_e\delta\Phi E + O_{2r}$$

(17i) $$\delta\Lambda = k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14}\alpha + a_{15}\beta + a_{16}\gamma + a_{17}s$$

(17ii) $$\delta\Phi = k_2 + a_{21}t_x + a_{22}t_y + a_{23}t_z + +a_{24}\alpha + a_{25}\beta + a_{26}\gamma + a_{27}s$$

The array elements of K and A are collected in the appendix.

(18) $$r = R + r_\varphi(k_2 + a_{21}t_x + a_{22}t_y + a_{23}t_z + a_{24}\alpha + a_{25}\beta + a_{26}\gamma + a_{27}s) + r_a\delta A + r_e\delta E$$

2.3 Datum Transformation of Universal Lambert Conic Project (ULC)-Cartesian form

The equations for datum transformation of ULC, Cartesian form, are given in equations (19) and (20) below. Note again that the array elements of K and A are collected in the appendix.

(19) $$\tilde{x} = \tilde{X} - r(\Phi)n\sin\tilde{\alpha}\delta\Lambda + r_\varphi\cos\tilde{\alpha}\delta\Phi + r_a\cos\tilde{\alpha}\delta A + (-r\sin\tilde{\alpha}\tilde{\alpha}_{e+}r_e\cos\tilde{\alpha})\delta\mathrm{E}$$

(20) $$\tilde{y} = \tilde{Y} + r(\Phi)n\cos\tilde{\alpha}\delta\Lambda + r_\varphi\sin\tilde{\alpha}\delta\Phi + r_a\sin\tilde{\alpha}\delta A + (r\sin\tilde{\alpha}\tilde{\alpha}_{e+}r_e\cos\tilde{\alpha})\delta\mathrm{E}$$

$$\tilde{x} = \tilde{X} - n\tilde{Y}(k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14}\alpha + a_{15}\beta + a_{16}\gamma + a_{17}s)$$

(21) $$r_\varphi(\Phi; A, E)\cos(n\Lambda)k_2 + a_{21}t_x + a_{22}t_y + a_{23}t_z + a_{24}\alpha + a_{25}\beta + a_{26}\gamma + a_{27}s) +$$

$$r_a(\Phi; E)\cos(n\Lambda)\delta\mathrm{A} + (r\sin(n\Lambda)\tilde{\alpha}_{e+} r_e \sin(n\Lambda)\delta\mathrm{E}$$

$$\tilde{y} = \tilde{Y} - n\tilde{X}(k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14}\alpha + a_{15}\beta + a_{16}\gamma + a_{17}s)$$

(22) $$r_\varphi(\Phi; A, E)\sin(n\Lambda)k_2 + a_{21}t_x + a_{22}t_y + a_{23}t_z + a_{24}\alpha + a_{25}\beta + a_{26}\gamma + a_{27}s) +$$

$$r_a(\Phi; E)\sin(n\Lambda)\delta\mathrm{A} + (r\cos(n\Lambda)\alpha_{e+} r_e \sin(n\Lambda)\delta\mathrm{E}$$

3. RESULTS OF NUMERICAL EXAMPLE

Assume we have measured the ellipsoidal coordinates of a point by means of $\{\Lambda, \Phi, \mathrm{H}\}$ for instance by satellite positioning technology of type GPS, GLONAS or others, it is to be noted that for the synthesis of the "*design vector **K***" as well as for the *design matrix A*" we need the global ellipsoidal height. Secondly we have to get information of the variation of the 7 datum parameters and the 2 form parameters, namely about the basic data which establish a local and a global UMP chart.

Table 1a lists the datum transformation parameters between a set of global coordinates Table 2) and the corresponding local coordinates (Table 3), while Table 1b lists their corresponding ellipsoidal parameters. Eastings and Northings computed from equations (1) and (2) are given in Table 4. To illustrate the numerical precision of the transformation equations developed in this paper, Eastings and Northings are again computed via equations (11) and (12), once without the second order terms (Table 5), and again including the second order terms (Table 6). The computed Root Mean Square (RMS) of the discrepancies in Table 5 for the Eastings and Northings are 0.0623m and 0.0127m respectively, indicating differences with the original data in centimeter range. On the other hand, the RMS discrepancies in Table 6 are 0.0049m and 0.0029m for Eastings and Northings respectively, showing differences in millimeter.

Table 1a: Datum transformation parameters (DTP)

DTP	t_x (m)	t_y (m)	t_z (m)	*a*	*b*	*g*	s
Values	450.911	60.121	-200.256	0".0578	0''.0366	-2''.396	-10.11 x 10^{-6}

Table 1b: Ellipsoidal parameters (EP)

EP	a (m)	e^2	A (m)	E^2	$\Phi_1 = \varphi_1$	$\Phi_2 = \varphi_2$
Values	6378388.00	0.00672267	6378137.00	0.00669438	51°10'00"	49°50'00"

Table 2: Global coordinates

Point No	Λ (deg. min. sec.)			Φ (deg. min. sec.)			H (m)
1	4	00	5.000	50	00	3.0000	1000.0000
2	4	00	3.000	50	03	2.0000	200.0000
3	5	00	1.000	50	12	1.0000	700.0000
4	3	54	7.000	50	26	0.0000	400.0000
5	4	30	0.000	50	22	8.0000	1000.0000
6	4	57	1.000	50	24	0.0000	1500.0000
7	5	27	0.000	50	36	7.0000	220.0000
8	5	51	0.000	50	36	0.0000	100.0000
9	5	27	30.000	50	37	4.0000	400.0000
10	4	27	3.7413	50	44	0.0000	500.0000
11	4	24	0.000	50	51	0.0000	300.0000
12	4	24	1.000	51	15	3.0000	200.0000
13	3	45	0.000	51	05	1.0000	700.0000
14	3	45	2.000	50	46	0.0000	800.0000

Table 3: Local coordinates

Point No	Λ (deg.	Min.	sec.)	Φ (deg.	Min.	sec.)
1	4	00	8.8981	49	59	50.5299
2	4	00	6.9001	50	02	49.5237
3	5	00	4.5060	50	11	484990
4	3	54	10.9533	50	27	47.4666
5	4	30	3.7104	50	21	55.4893
6	4	57	4.5308	50	23	47.8224
7	5	27	3.3348	50	35	54.4564
8	5	51	3.1736	50	35	47.4532
9	5	27	33.3317	50	36	51.4552
10	4	27	3.7414	50	43	47.4549
11	4	24	3.7650	50	50	47.4443
12	4	24	4.7772	51	14	50.4070
13	3	45	4.0363	51	04	48.4321
14	3	45	6.0246	50	45	47.4617

Table 4: Eastings and Northnings computed from equations (1) and (2)

Point No	Easting (m)	Northing (m)
1	5316191.79574	286842.27740
2	5310671.56651	286504.61547
3	5289719.58411	356830.61547
4	5264828.67860	277000.86058
5	5273283.25828	320050.72324
6	5267793.00906	351788.79538
7	5242904.40447	385581.98699
8	5240967.14964	413834.06750
9	5241104.63748	314071.68189
10	5233031.95125	314071.68189
11	5220287.09258	309779.45390

12	5175772.27391	307157.54755
13	5196843.38468	262762.71426
14	5232054.46840	264582.07354

Table 5: Eastings and Northings computed from equations (11) and (12) *without the second order terms*, and their differences (δ E, δ N) with that of table 4.

Point No	Easting(m)	Northing(m)	δ E(m)	δ N(m)
1	5316191.73310	286842.26461	-.06264	-.01279
2	5310671.50437	286504.60277	-.06214	-.01270
3	5289719.52240	356830.63355	-.06171	-.01271
4	5264828.61644	277000.84813	-.06216	-.01245
5	5273283.19744	320050.71153	-.06084	-.01171
6	5267793.94685	351788.78227	-.06221	-.01311
7	5242904.34270	385581.97431	-.06177	-.01268
8	5240967.08702	413834.05540	-.06262	-.01210
9	5241104.57533	314071.95256	-.06215	-.01174
10	5233031.88781	314071.66870	-.06344	-.01319
11	5220287.02961	309779.44159	-.06297	-.01231
12	5175772.21001	307157.53512	-.06390	-.01243
13	5196843.32121	262762.70049	-.06347	-.01377
14	5232054.40769	264582.06021	-.06071	-.01333

Table 6: Eastings and Northings computed from equations (11) and (12) *including the second order terms,* and their differences (δ E, δ N) with that of Table 4.

Point No	Easting(m)	Northing(m)	δ E(m)	δ N(m)
1	5316191.79052	286842.27479	-.00522	-.00261
2	5310671.56180	286504.61296	-.00471	-.00251
3	5289719.57989	356830.61296	-.00422	-.00323
4	5264828.67391	277000.85843	-.00469	-.00215
5	5273283.25492	320050.72138	-.00336	-.00186
6	5267793.00436	351788.79179	-.00470	-.00359
7	5242904.40026	385581.98348	-.00421	-.00351
8	5240967.14461	413834.06427	-.00503	-.00323
9	5241104.63290	314071.67861	-.00458	-.00257
10	5233031.94533	314071.45155	-.00592	-.00328
11	5220287.08715	309779.45390	-.00543	-.00235
12	5175772.26759	307157.54510	-.00632	-.00245
13	5196843.37874	262762.71094	-.00594	-.00332
14	5232054.46519	264582.07064	-.00321	-.00290

4. CONCLUSION

This paper has illustrated a new method of transformation of conformal coordinates of type Lambert Conic from a global datum to a local datum, and vise versa. The method takes advantage of Taylor expansion of coordinate differences and differences in ellipsoidal form parameters up to second order terms.

It is observed that in order to achieve millimeter precision in the computation of transformed coordinates, the second order terms of the transformation equations should not be neglected.

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BIOGRAPHICAL NOTES

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APPENDIX

Curvilinear Datum Transformation

Points on the *Earth surface* are described by Cartesian coordinates {x,y,z), {X,Y,Z} of the embedded space assumed to be a *Weitzenboeck vector space of three* dimensions with respect to the center of mass of the Earth, labeled $\underset{\sim}{O}$ as the origin and the principal axis of an ellipsoid of revolution $E_{a,b}, E^2_{\mathrm{A,B}}$ in a local *or* global frame of reference. The two sets {x,y,z}, {X,Y,Z} are related by a rectangular datum transformation which leaves as a passive transformation angles and distance ratios equivariant ("invariant"). In its linear variant, the seven-parameter transformation which leaves angles and distance ratios equivariant is the *conformal group* $C_7(3)$ is parameterised, "closed to the *identity*", by three parameters $\{t_x,t_y,t_z\}$of *translation*, three parameters $\{\alpha,\beta,\gamma\}$of *rotations* and one *scale* parameter s. With the forward and backward transformation close to identity of conformal type, the final form (A1) is gained.

If we perturb local "surface normal" ellipsoidal longitude λ, ellipsoidal latitude φ, and ellipsoidal height *h* by means of (A14), $\lambda = \Lambda + \partial\Lambda$, $\varphi = \Phi + \partial\Phi$, $h = H + \partial H$ we are lead to the Taylor expansion $\{ x(\Lambda + \partial\Lambda, \Phi + \partial\Phi, H + \partial H)$, $y(\Lambda + \partial\Lambda, \Phi + \partial\Phi, H + \partial H)$, $z(\Lambda + \partial\Lambda, \Phi + \partial\Phi, H + \partial H) \}$, namely to (A4). The *zero order vector-valued term* j contains the effect of different form parameters $\{ a, e^2 \}$, $(A, E^2 \}$, respectively. In contrast the *first order matrix-values term* J represent in a first approximation the influence of the variation in ellipsoidal coordinates {Λ,Φ,H}, {λ,φ,h}, respectively.

A first remark is concerned with the functional representation h(λ,φ), H(Λ,Φ), respectively. Since the curvilinear coordinates represent the earth surface conventionally considered to be a two–dimensional Riemann manifold, two parameters/two coordinates, namely (λ,φ), (Λ,Φ) respectively are sufficient to coordinate points on the earth surface. Accordingly the height (*H, h*) must be functions H(Λ, Φ), h(λ, φ) of the surface parameters/surface coordinates (Λ,Φ), (λ,φ), respectively. These functions of type orthonormal on $E^2_{\mathrm{a,b}}$ or $E^2_{\mathrm{A,B}}$, respectively, have been numerically given in *Grafarend and Engels (1992).*

A second remark refers to the zero terms of the Taylor expansion of the function:

$$x(\lambda, \varphi, h; a, e^2) = x(\Lambda + \partial\Lambda, \Phi + \partial\Phi, H + \partial H; a, e^2)$$
$$y(\lambda, \varphi, h; a, e^2) = y(\Lambda + \partial\Lambda, \Phi + \partial\Phi, H + \partial H; a, e^2)$$
$$z(\lambda, \varphi, h; a, e^2) = z(\Lambda + \partial\Lambda, \Phi + \partial\Phi, H + \partial H; a, e^2)$$

In the previous contributions *Grafarend, E , Krumm, F., and F. Okeke (1995),* and *Grafarend, E. and F. Okeke (1998)* we have linearized those functions in terms of $a = A + \partial A$, $e^2 = E^2 + \partial E^2$, namely with Jacobi terms $j_A \partial A + j_{E^2} \partial E^2$, too! Here, as well as in *Grafarend and R. Syffins (1998)* we have refrained from such an additional linearization for numerical reasons. A gain in numerical precision is achieved without linarization, or in order words, linearization in terms of ($\partial A, \partial E^2$) is not accurate enough.

Let us go back to the *"direct curvilinear datum transformation"* (A6), which is achieved in the following way. The Taylor expansion leads to the Cartesian coordinate increments (A4) of type

$$
\begin{aligned}
x - X &= f_x(\lambda - \Lambda, \varphi - \Phi, h - H; a, e^2, A, E^2) \\
y - Y &= f_y(\lambda - \Lambda, \varphi - \Phi, h - H; a, e^2, A, E^2) \\
z - Z &= f_z(\lambda - \Lambda, \varphi - \Phi, h - H; a, e^2, A, E^2) \,,
\end{aligned}
$$

namely in the linearized form in terms of $((\lambda - \Lambda, \varphi - \Phi, h - H)$. The solution of the system of equations (A4) is (A5) which relates

$$
\begin{aligned}
\lambda - \Lambda &= g_\lambda(x - X, y - Y, z - Z, a, e^2, A, E^2) \\
\varphi - \Phi &= g_\varphi(x - X, y - Y, z - Z, a, e^2, A, E^2) \\
h - H &= g_h(x - X, y - Y, z - Z, a, e^2, A, E^2) \,,
\end{aligned}
$$

namely in the linearized form in terms of $(x - X, y - Y, z - Z)$. Finally by means of the "*direct Cartesian datum transformation*" close to the identity, namely (A1), inserted into (A5) as quoted above we arrive at the destination point the "*direct curvilinear datum transformation*" close to the identity, namely (A6). The influence of the difference in form parameters (a, e^2), (A, E^2), respectively is to be seen in the transformed form parameter vector $K = J^{-1} j$, j given in equation (A9). In contrast, the 7 parameters of the conformal group $C_7(3)$ of type translation (t_x, t_y, t_z), rotation (α, β, γ) and scale s act on the datum parameter Jacobi matrix $A = J^{-1} S$, J^{-1} given in equation (A10), namely based on the explicit form j, J^{-1} reviewed in equations (A7) and (A8).

The conformal group $C_7(3)$, forward and backward transformation, Cartesian coordinates $\{ x, y, z \}$ of local type versus Cartesian coordinates $\{ X, Y, Z \}$ of the global type covering $\mathbb{R}^3$ equipped with an Euclidean matrix is given as follows.

$$
\begin{bmatrix} x - X \\ y - Y \\ z - Z \end{bmatrix} = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & -Z & Y & X \\ 0 & 1 & 0 & +Z & 0 & -Z & Y \\ 0 & 0 & 1 & -Y & +Z & 0 & Z \end{array}\right] \begin{bmatrix} t_x \\ t_y \\ \frac{t_x}{\alpha} \\ \beta \\ \frac{\gamma}{s} \end{bmatrix} \qquad \text{(A1i)}
$$

$$\begin{bmatrix} x-X \\ y-Y \\ z-Z \end{bmatrix} = S \begin{bmatrix} t_x \\ t_y \\ \frac{t_x}{\alpha} \\ \beta \\ \frac{\gamma}{s} \end{bmatrix} \forall S \in R^{3\times 7} \qquad \text{(A1ii)}$$

The corresponding Curvilinear backward – forward transformation is then given as follows

$$\begin{bmatrix} x-X \\ y-Y \\ z-Z \end{bmatrix} = \begin{bmatrix} [n(\varphi)+h(\lambda,\varphi)]\cos\varphi\cos\lambda - [N(\Phi)+H(\Lambda,\Phi)]\cos\Phi\cos\Lambda \\ [n(\varphi)+h(\lambda,\varphi)]\cos\varphi\sin\lambda - [N(\Phi)+H(\Lambda,\Phi)]\cos\Phi\sin\Lambda \\ [(1-e^2)n(\varphi)+h(\lambda,\varphi)]\sin\varphi - [(1-E^2)N(\Phi)+H(\Lambda,\Phi)]\sin\Phi \end{bmatrix} \qquad \text{(A2)}$$

$$\lambda = \Lambda + \delta\Lambda,\ \varphi = \Phi + \delta\Phi,\ h = H + \delta H,\ a = A + \delta A,\ e^2 = E^2 + \delta E^2. \qquad \text{(A3)}$$

Applying Taylor expansion we have

$$\begin{bmatrix} x-X \\ y-Y \\ z-Z \end{bmatrix} = j + J \begin{bmatrix} \lambda-\Lambda \\ \varphi-\Phi \\ h-H \end{bmatrix} + O_2(\delta\lambda^2, \delta\varphi^2, \delta h^2) \qquad \text{(A4)}$$

$$\begin{bmatrix} \lambda-\Lambda \\ \varphi-\Phi \\ h-H \end{bmatrix} = -J^{-1} j + J^{-1} \begin{bmatrix} x-X \\ y-Y \\ z-Z \end{bmatrix} + O_2(\delta\lambda^2, \delta\varphi^2, \delta h^2) \qquad \text{(A5)}$$

The vector j and the matrix J^{-1} are given in equations (A7) and (A8) respectively. The direct curvilinear datum transformation is then given by

$$\{ \begin{bmatrix} \delta\Lambda \\ \delta\Phi \\ \delta H \end{bmatrix} = \begin{bmatrix} \lambda-\Lambda \\ \varphi-\Phi \\ h-H \end{bmatrix} = K + A \begin{bmatrix} t_x \\ t_y \\ t_z \\ \alpha \\ \beta \\ \gamma \\ s \end{bmatrix}; K = -J^{-1} j \in R^{3x7} \} \qquad \text{(A6)}$$

$$j = \begin{bmatrix} j_x \\ j_y \\ j_x \end{bmatrix} = \begin{bmatrix} \left(\frac{a}{\sqrt{1-e^2\sin^2\Phi}} - \frac{A}{\sqrt{1-E^2\sin^2\Phi}}\right)\cos\Phi\cos\Lambda \\ \left(\frac{a}{\sqrt{1-e^2\sin^2\Phi}} - \frac{A}{\sqrt{1-E^2\sin^2\Phi}}\right)\cos\Phi\sin\Lambda \\ \left(\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\Phi}} - \frac{A(1-E^2)}{\sqrt{1-E^2\sin^2\Phi}}\right)\sin\Phi \end{bmatrix} \quad \text{(A7)}$$

The datum parameter Jacobi matrix, J is the matrix of partial derivates of x, y, z, with respect to j ,l ,h. The matrix J^{-1} is derived from J and is given as

$$J^{-1} = \begin{bmatrix} -\frac{\sin\Lambda}{(n(\Phi)+H)\cos\Phi} & \frac{\cos\Lambda}{(n(\Phi)+H)\cos\Phi} & 0 \\ -\frac{\cos\Lambda\sin\Phi}{m(\Phi)+H} & -\frac{\sin\Phi\sin\Lambda}{m(\Phi)+H} & \frac{\cos\Phi}{m(\Phi)+H} \\ \cos\Phi\cos\Lambda & \cos\Phi\sin\Lambda & \sin\Phi \end{bmatrix}. \quad \text{(A8)}$$

Given that

$$P(\Phi) = \frac{\left[a(1-E^2\sin^2\Phi)^{1/2}\right]}{\left[a(1-E^2\sin^2\Phi)^{1/2}\right]}; \quad Q(\Phi) = \frac{\left[a(1-e^2)(1-E^2\sin^2\Phi)^{3/2}\right]}{\left[A(1-E^2)(1-e^2\sin^2\Phi)^{3/2}\right]}.$$

Then

$$K := J-1j = \begin{bmatrix} k1 \\ k2 \\ k3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sin\Phi\cos\Phi}{QM+H}(Pe^2-E^2) \\ N\left[P-1-\sin^2\Phi(Pe^2-E^2)\right] \end{bmatrix} \quad \text{(A9)}$$

$$A := J^{-1}S = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \end{bmatrix} \quad \text{(A10i)}$$

$$\begin{cases} a_{11} = -\sin\Lambda / \left[(PN+H)\cos\Phi\right] \\ a_{12} = +\cos\Lambda / \left[(PN+H)\cos\Phi\right] \\ a_{13} = 0 \\ a_{14} = \left[(1-E^2)PN+H\right]\sin\Phi\cos\Lambda / \left[(N+H)\cos\Phi\right] \\ a_{15} = \left[(1-E^2)PN+H\right]\sin\Phi\sin\Lambda / \left[(N+H)\cos\Phi\right] \\ a_{16} = -(N+H)/(PN+H) \\ a_{17} = 0 \end{cases} \qquad \text{(A10ii)}$$

$$\begin{cases} a_{21} = -\sin\Phi\cos\Lambda/(QM+H) \\ a_{22} = -\sin\Phi\sin\Lambda/(QM+H) \\ a_{23} = \cos\Phi/(QM+H) \\ a_{24} = -\left[N^2(1-E^2)+MH\right]\sin\Lambda / \left[M(QM+H)\right] \\ a_{25} = \left[N^2(1-E^2)+MH\right]\cos\Lambda / \left[M(QM+H)\right] \\ a_{26} = 0 \\ a_{27} = -E^2 N\sin\Phi\cos\Phi/(QM+H) \end{cases} \qquad \text{(A10iii)}$$

$$\begin{cases} a_{31} = \cos\Phi\cos\Lambda \\ a_{32} = \sin\Phi\sin\Lambda \\ a_{33} = \sin\Phi \\ a_{34} = -NE^2\sin\Phi\cos\Phi\sin\Lambda \\ a_{35} = NE^2\sin\Phi\cos\Phi\cos\Lambda \\ a_{36} = 0 \\ a_{27} = H + N(1-E^2\sin^2\Phi) \end{cases} \qquad \text{(A10iv)}$$