Vertical Reference Frames in Practice – Introduction & Definitions

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Modernising Height Systems – Why Now?

1) The increasing use of geometric GNSS techniques motivates development of means to determine heights from GNSS related to a local vertical datum
2) Physical heights H have traditionally been obtained using terrestrial optical levelling techniques... *but notions of "uphill-and-downhill", and direction of gravity, remain important for many users of height systems*
3) Standard optical levelling is expensive, laborious and time-consuming... *in addition, it is difficult in remote and mountainous areas and the inherent systematic errors grow very quickly over large distances*
4) On the other hand, h from GNSS can be obtained quickly and inexpensively, and converted to H using geoid height N values available from the international geodetic community or from mapping agencies
5) High expectations of continued improvement in determination of N values from global and regional geoid models
6) We need a closer analysis of the means of determining H , h and N (and their uncertainties)

8) The gravity field model information derived from CHAMP, GRACE and GOCE has main advantages, such as global consistency, high accuracy geoid height, etc... *although over wavelengths of 100km or more*
9) Fine resolution geoid information N can be derived from the processing of airborne gravimetry data
10) The new high quality global gravity field models, together with the high accuracy of the geometrical reference system (ITRF) and improved geometrical sea surface mapping via satellite altimetry, allows for "absolute" height datum definition... *and hence the possibility of defining an International Vertical Reference Frame (IVRF)*
11) Increasing interest in determining offsets of local vertical datums with respect to a modern consistent vertical reference frame
12) But there are some unique issues related to height systems and datums that need to be addressed... *"Vertical Reference Frame in Practice"*

Reference Systems, Frames, Surfaces & Datums

In general:

Reference systems are needed to describe the Earth's **geometry** and **gravity field** as well the Earth's **orientation** in space

A reference system defines constants, conventions, models, and parameters required for the mathematical representation of **geometric** and **physical** quantities

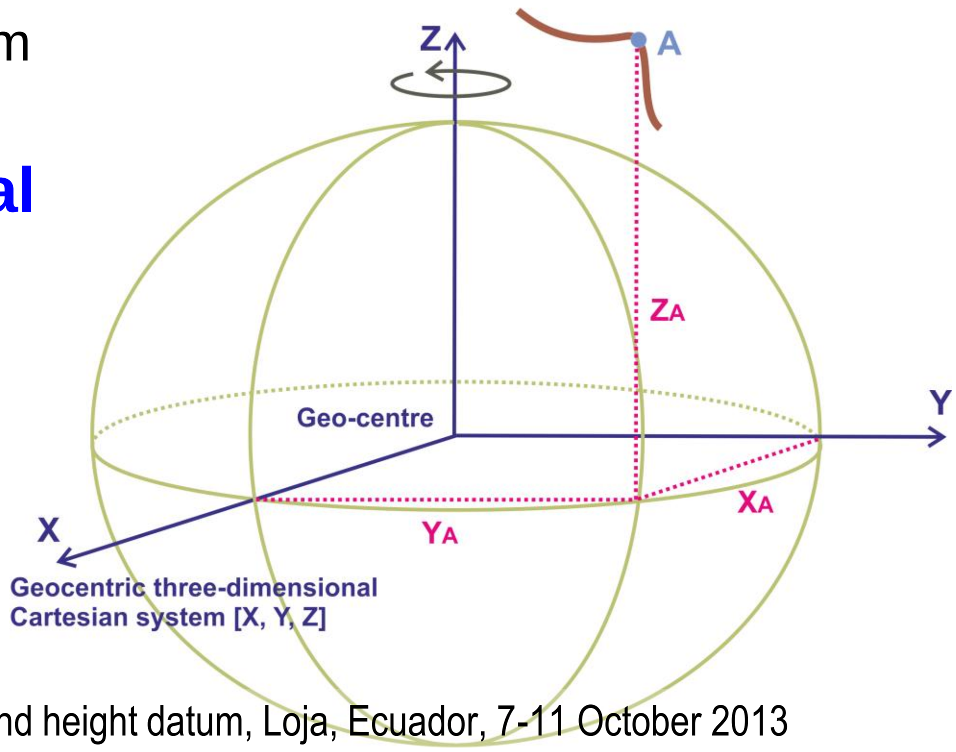
L. Sanchez, 11th Int. School of the Geoid Service: heights and height datum, Loja, Ecuador, 7-11 October 2013

A reference frame realises a reference system in two ways:

- **physically**, by a set of points, marks or observatories
- **mathematically**, by the determination of coordinates (or other geodetic quantities)

  referring to that reference system

The realisation of a **conventional reference system** is a **conventional reference frame**, e.g. International Terrestrial Reference Frame (ITRF)



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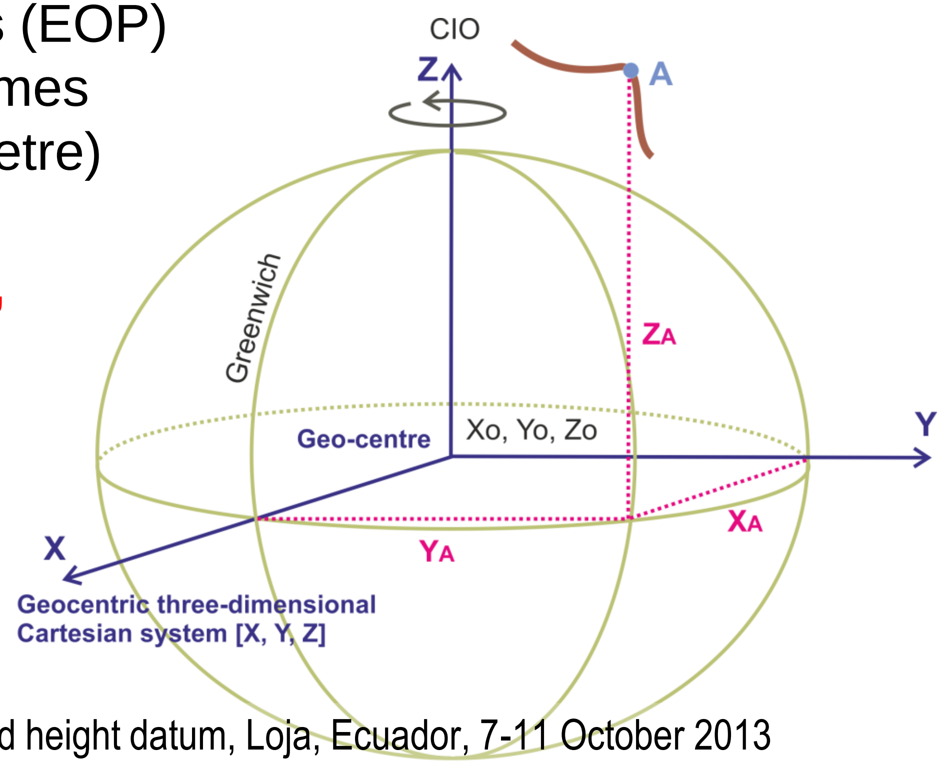
The datum fixes the relation between a reference frame and a reference system. e.g. for the ITRS:

- (implied) coordinates of the geocentre X_0,Y_0,Z_0
- direction of the coordinate axes (EOP) linking terrestrial & celestial frames
- scale (unit of length, e.g. the metre)

In general, the words "datum" and "frame" may be considered synonymous



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TOPOGRAPHIC SURFACE

- Physical surface of the Earth for land and sea bottom
- **"The actual figure of the Earth"**
- **Most survey measurements are made on land surface**

GEOID SURFACE

- Equipotential surface of the Earth's gravity field which best fits (global) MSL
- **"The simplified figure of the Earth"**
- Due to the uneven distribution of the Earth's mass, the geoidal surface is irregular
- The direction of gravity is perpendicular to the geoid at a point and is the **"vertical"** or **"plumbline"**
- **Traditional surveying instruments are sensitive to the geoid and gravity**

H.K. Lee, Vertical Height Systems – Asia-Pacific Region, FIG Congress, KL, Malaysia, 14-21 June 2014

Ellipsoidal Surface

- **Simple mathematical model of the Earth** which closely approximates **the geoid**
- Computation of horizontal geodetic coordinates is performed on the surface
- GNSS coordinates can be converted to quantities that relate to the ellipsoid
- A Reference Ellipsoid is a conventionally defined model surface, e.g. GRS80
- A line perpendicular to the surface of the ellipsoid at a point on it is **"normal"**
- Angle between the vertical and the normal is the **"deflection of the vertical"**

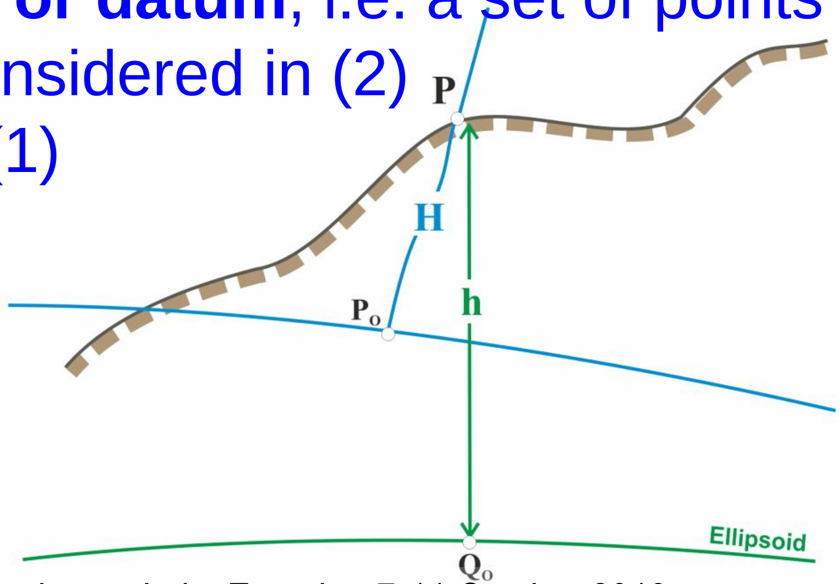
Sea Level Surface

- **MSL is an approximation to the geoid**
- Sea level (SL) varies due to tides, currents, SLR, etc
- Deviation between SL and geoid is the **"sea surface topography"**
- Tide gauges can be used to derive MSL at specific coastal points
- Satellite altimetry can geometrically map the instantaneous SL wrt the ellipsoid
- A knowledge of the static and time-varying SL is important for geodesy & oceanography

A **vertical reference system** is basically composed of:

1) **a reference surface**, i.e. the zero-height level (or **vertical datum**)
2) **a vertical coordinate**, i.e. the “type of height”

Its realisation is a **vertical network or datum**, i.e. a set of points whose heights, of the same type considered in (2) and refer to the datum specified in (1)



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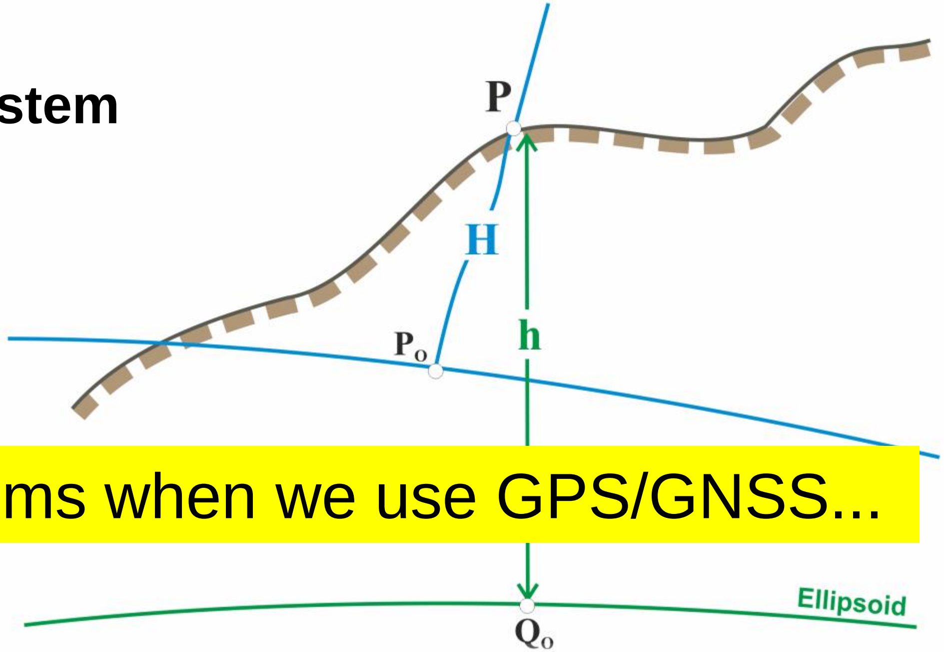
Vertical Coordinate Types

If the *height type* and *reference surface* depend on the Earth's gravity field, we are dealing with a **physical height system** (e.g. **orthometric heights** and **geoid**, or **normal heights** and **quasi-geoid**)

If not, it is a **geometrical height system** (e.g. **ellipsoidal heights** and **reference ellipsoid**)



We must deal with both systems when we use GPS/GNSS...

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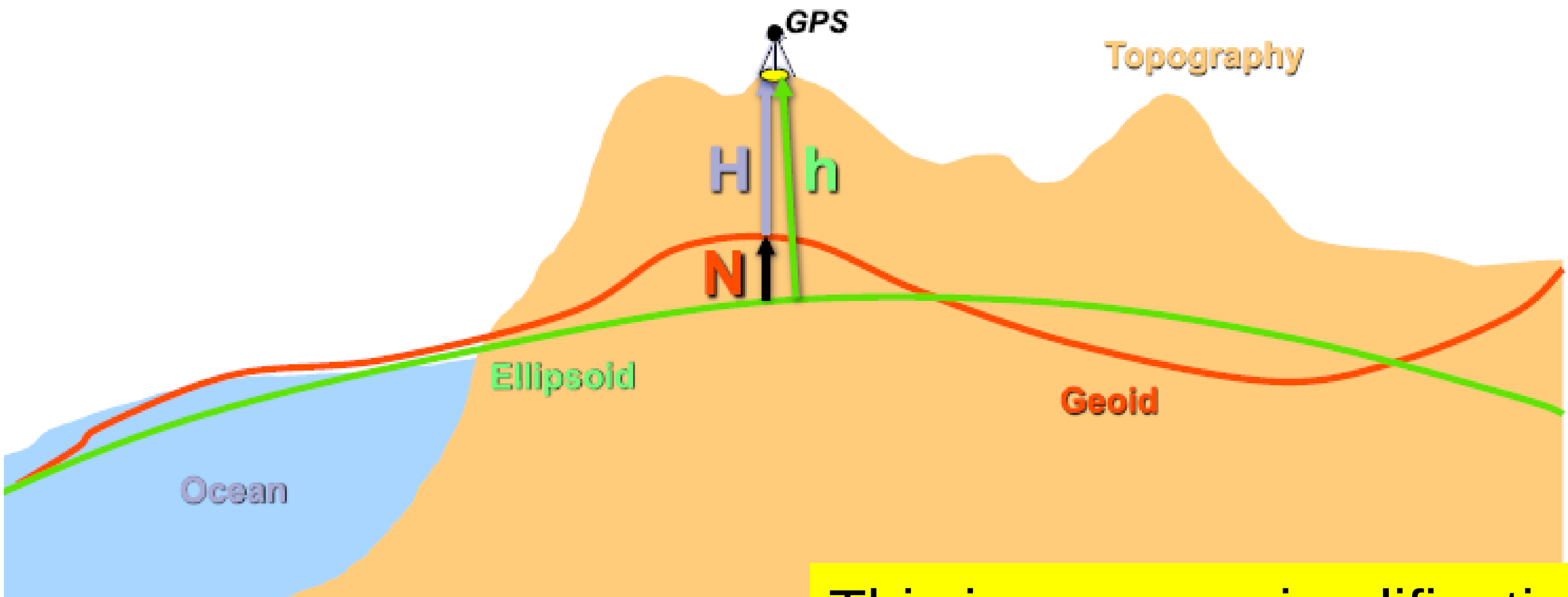
Putting it all together...



This is a gross simplification...

$$H_{\text{(from leveling)}} = h_{\text{(from GPS)}} - N_{\text{(height of the geoid)}}$$

J. Agren, Gravity & Height for National Mapping & Geodetic Surveying, Dublin, Ireland, 2-6 February 2015

Ellipsoidal (Geometric) Height System

Linear distance of a point P above (or below) the Reference Ellipsoid (RE), measured along the normal to the RE

- **Ellipsoidal heights** are unique to the RE, **directly related to the size, shape, location & orientation of the RE**
- Ellipsoidal heights have no relation to the gravity field of the Earth, being purely geometric in concept and **cannot define the direction of water flow**
- **Ellipsoidal height is derived using GNSS**, and is the quantity (together with ϕ,λ) in the conversion to/from X,Y,Z

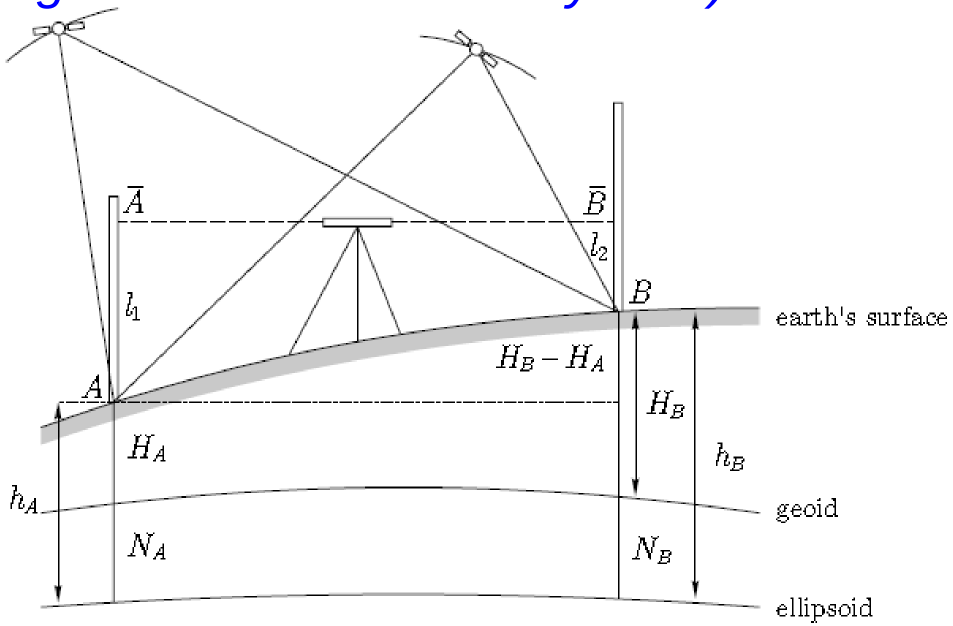
http://en.wikipedia.org/wiki/Geodetic_system#Vertical_datum

Orthometric (Physical) Height System

Distance between geoid and point P, measured along the curved plumbline (the vertical) through P

- **Orthometric height** (OH) is independent of the RE
- OH can be derived from optical **height difference observations**, *in a number of different ways (see later slides)*
- Directly affected by the gravity field and **useful for engineering**, e.g. water flow, as OH values can be used to define "level" surfaces
- In principle, **orthometric height** (H) can be related to the **ellipsoidal height** (h) of a point through the geoid height (N): $H = h - N$

- Spirit levelling or trigonometrical heighting: Gives **physical** height differences defined in the gravity field. *Refers to qeoid (or quasi-geoid)*
- Height determination by GNSS gives **geometric** heights (or height differences) independently of the gravity field. *Refers to Reference Ellipsoid (or equivalently wrt a 3D geocentric Cartesian system)*



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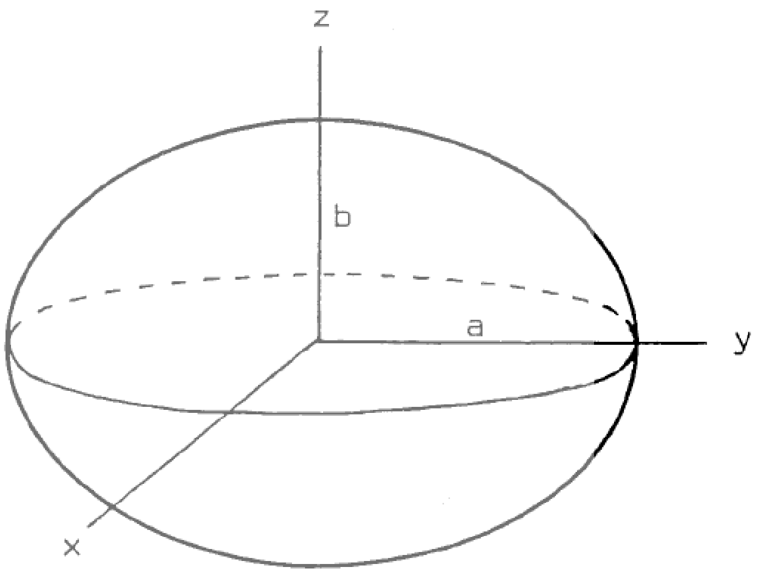
Some Definitions: Reference Ellipsoid

Reference Ellipsoid

- Rotational ellipsoid:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

- "Geometrical model of the Earth"...
  Size & shape defined by two parameters
- Physically centred at the **geocentre**
- Centre is origin of 3D Cartesian coordinate frame
- **Approximated by the geoid and MSL to within +/-100m**



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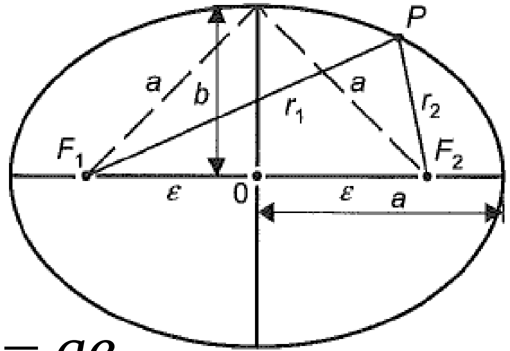
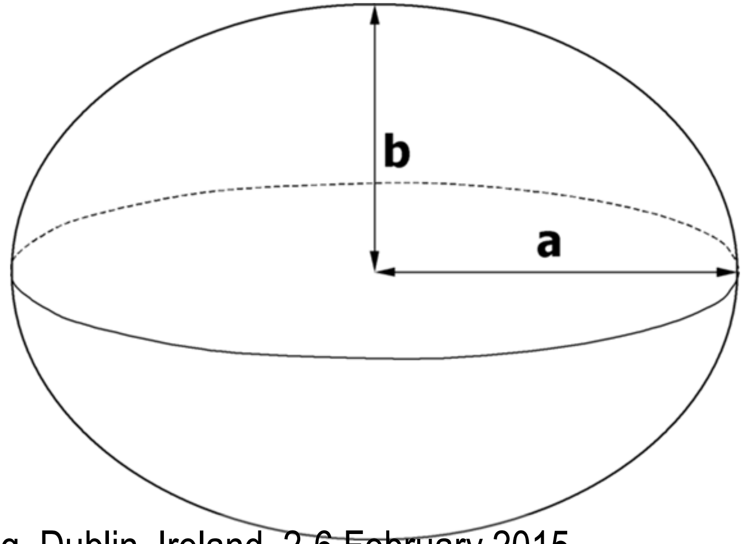
- **The rotational ellipsoid is uniquely defined by 2 parameters**, for instance:
  - Semi major axis *a* + semi minor axis *b*
  - Semi major axis *a* + flattening *f*
  - Semi major axis *a* + first eccentricity e



$$\varepsilon = ae$$

- Flattening: $$f = \frac{a-b}{a}$$



- First eccentricity: $$e = \frac{\sqrt{a^2 - b^2}}{a}$$

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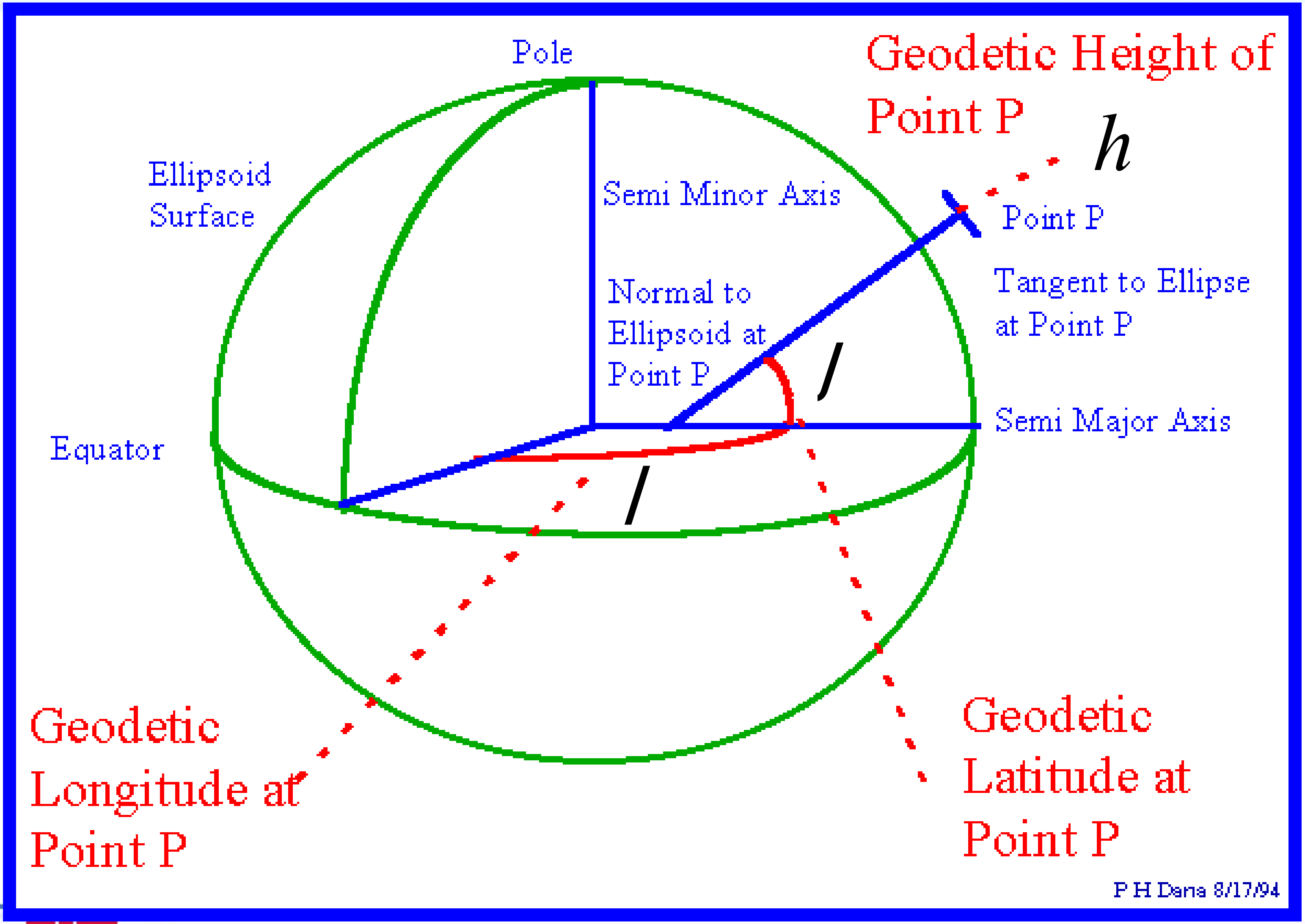
Commonly Used Reference Ellipsoids

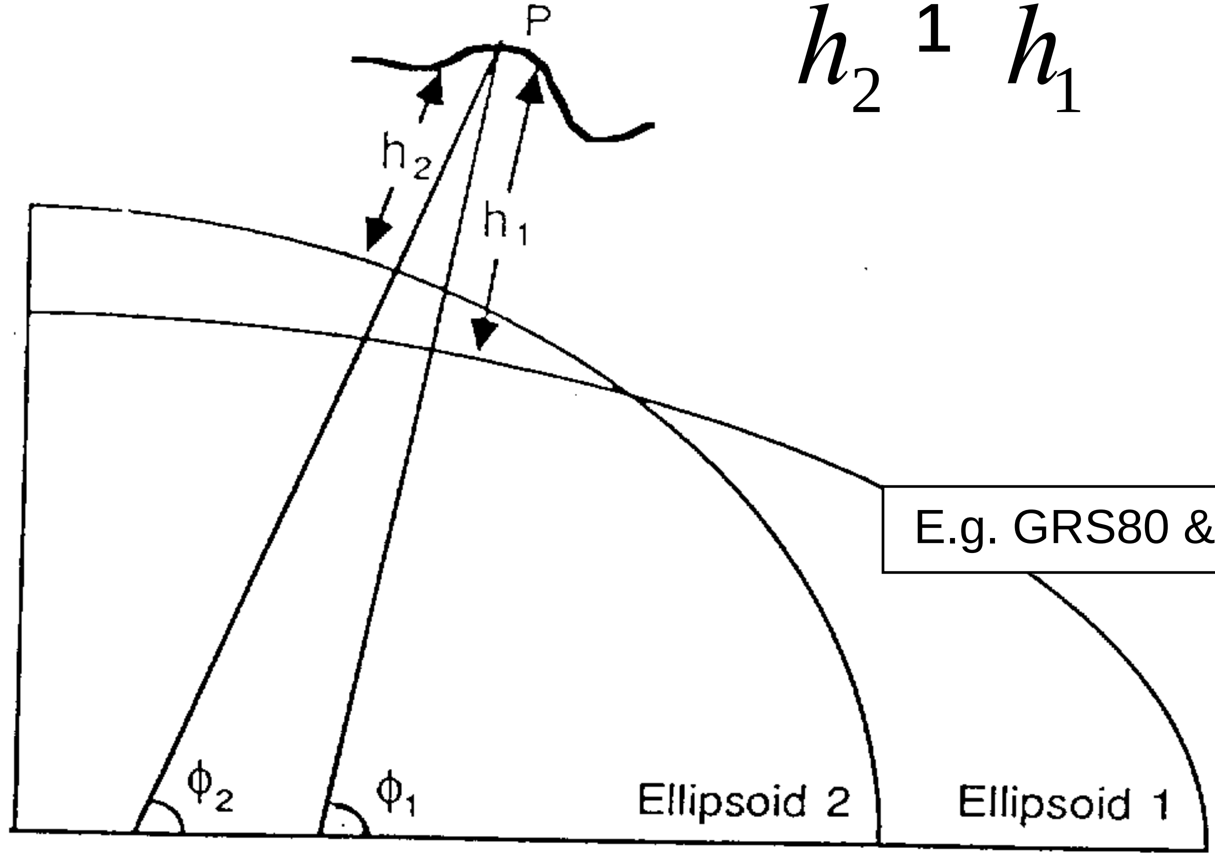
Ellipse	Semi-Major Axis (meters)	1/Flattening
Airy 1830	6377563.396	299.3249646
Bessel 1841	6377397.155	299.1528128
Clarke 1866	6378206.4	294.9786982
Clarke 1880	6378249.145	293.465
Everest 1830	6377276.345	300.8017
Fischer 1960 (Mercury)	6378166.0	298.3
Fischer 1968	6378150.0	298.3
G R S 1967	6378160.0	298.247167427
G R S 1975	6378140.0	298.257
G R S 1980	6378137.0	298.257222101
Hough 1956	6378270.0	297.0
International	6378388.0	297.0
Krassovsky 1940	6378245.0	298.3
South American 1969	6378160.0	298.25
WGS 60	6378165.0	298.3
WGS 66	6378145.0	298.25
WGS 72	6378135.0	298.26
WGS 84	6378137.0	298.257223563

Peter H. Dana 9/1/94


Pole
Geodetic Height of Point P
h
Ellipsoid Surface
Semi Minor Axis
Point P
Normal to Ellipsoid at Point P
Tangent to Ellipse at Point P
Semi Major Axis
Equator
Geodetic Longitude at Point P
Geodetic Latitude at Point P
P H Dana 8/17/94

Effect of Different Reference Ellipsoids on Ellipsoidal Height

h_2 1 h_1



E.g. GRS80 & ANS REs

<---23m--->

Geodetic Reference System 1980 (GRS80)

- Parameters of the GRS 80 are used for the standard Reference Ellipsoid underpinning most (horizontal) datums, and for defining the **normal gravity field**
- The **four defining parameters** are:

Parameter and value	Description
$a = 6\,378\,137$ m	semimajor axis of the ellipsoid
$GM = 3\,986\,005 \cdot 10^8\ \mathrm{m^3\,s^{-2}}$	geocentric gravitational constant of the earth (including the atmosphere)
$J_2 = 108\,263 \cdot 10^{-8}$	dynamical form factor of the earth (excluding the permanent tidal deformation)
$\omega = 7\,292\,115 \cdot 10^{-11}\ \mathrm{rad\,s^{-1}}$	angular velocity of the earth

- From these, **many other parameters can be derived**:

Parameter and value	Description
Geometrical constants	
$b = 6\,356\,752.3141$ m	semiminor axis of the ellipsoid
$E = 521\,854.0097$ m	linear eccentricity
$c = 6\,399\,593.6259$ m	polar radius of curvature
$e^2 = 0.006\,694\,380\,022\,90$	first eccentricity squared
$e'^2 = 0.006\,739\,496\,775\,48$	second eccentricity squared
$f = 0.003\,352\,810\,681\,18$	flattening
$1/f = 298.257\,222\,101$	reciprocal flattening
Physical constants	
$U_0 = 62\,636\,860.850\ \mathrm{m^2\,s^{-2}}$	normal potential at the ellipsoid
$J_4 = -0.000\,002\,370\,912\,22$	spherical-harmonic coefficient
$J_6 = 0.000\,000\,006\,083\,47$	spherical-harmonic coefficient
$J_8 = -0.000\,000\,000\,014\,27$	spherical-harmonic coefficient
$m = 0.003\,449\,786\,003\,08$	$m = \omega^2 a^2 b/(GM)$
$\gamma_a = 9.780\,326\,7715\ \mathrm{m\,s^{-2}}$	normal gravity at the equator
$\gamma_b = 9.832\,186\,3685\ \mathrm{m\,s^{-2}}$	normal gravity at the pole

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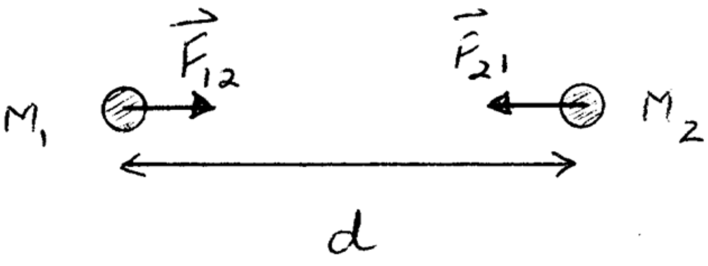
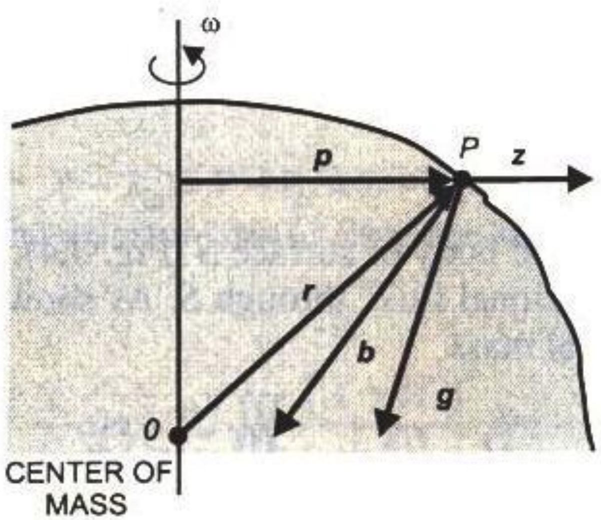
Some Definitions: Gravity, Geopotential & the Geoid

What is Gravity?





- **Newton's Law of Gravitation**:

$$F = G\frac{M_1 M_2}{d^2}$$

Gravity = gravitational force + centrifugal force

In Geodesy we study the force affecting a unit mass, i.e. acceleration

- Unit m/s^2 or Gal ($1\ m/s^2 = 100$ Gal)

9.81456789 m/s^2 = 981.456789 Gal
= 981 456.789 mGal
= 981 456 789 μGal

- Relation between the **gravity vector** and the **geopotential** *W:* $\vec{g} = \nabla W = grad\left(W\right)$

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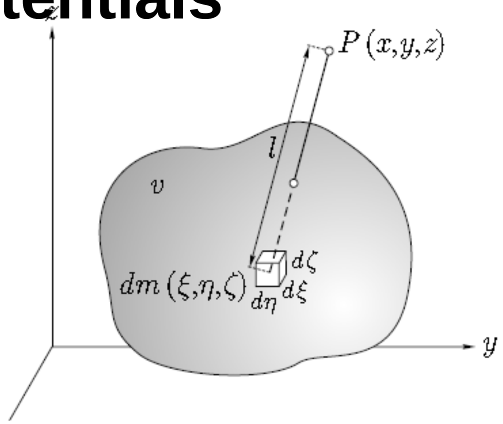
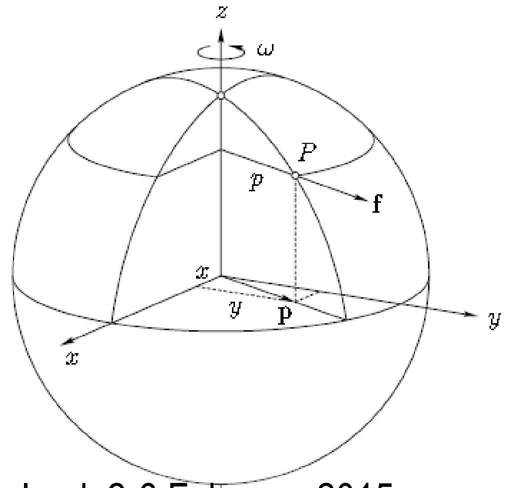
Gravitational, Centrifugal and Gravity Potentials

$$W = V + \Phi = G \iiint_v \frac{\rho}{l} dv + \frac{1}{2}\omega^2 \left(x^2 + y^2\right)$$



W	Gravity potental
V	Gravitational potential
G	Newton's gravitational constant
v	Density of the Earth's masses
l	Distance between the computation point and the mass element
Φ	Centrifugal potential
ω	Earth's rotational velocity
$p = \sqrt{x^2 + y^2}$	Distance from the rotation axis



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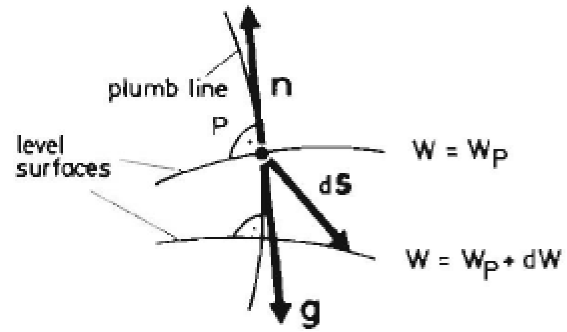
The gravity vector **g** (in Cartesian coordinates) and gravity g

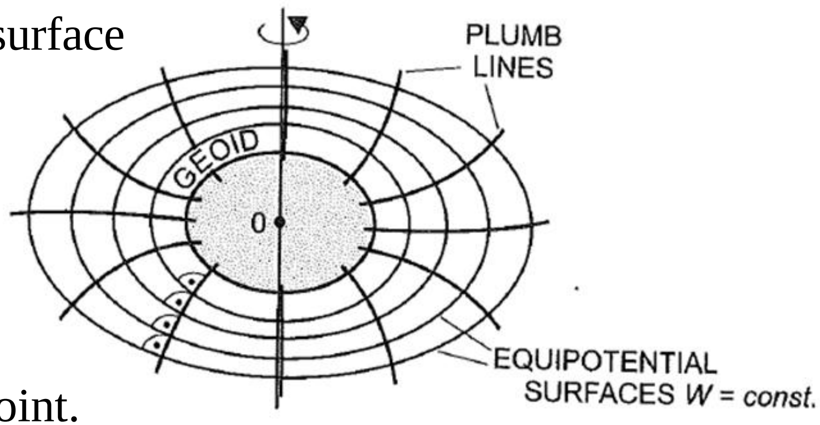
$$\mathbf{g} = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z} \right)^T \qquad g = |\mathbf{g}| = \left| \frac{\partial W}{\partial n} \right|$$



Equipotential surfaces (or level surfaces)

Surfaces where $W = const$

$$dW = \mathbf{g} \cdot d\mathbf{s} = g \cdot ds \cdot \cos(\mathbf{g}, d\mathbf{s}) = 0$$ for d**s** along a level surface

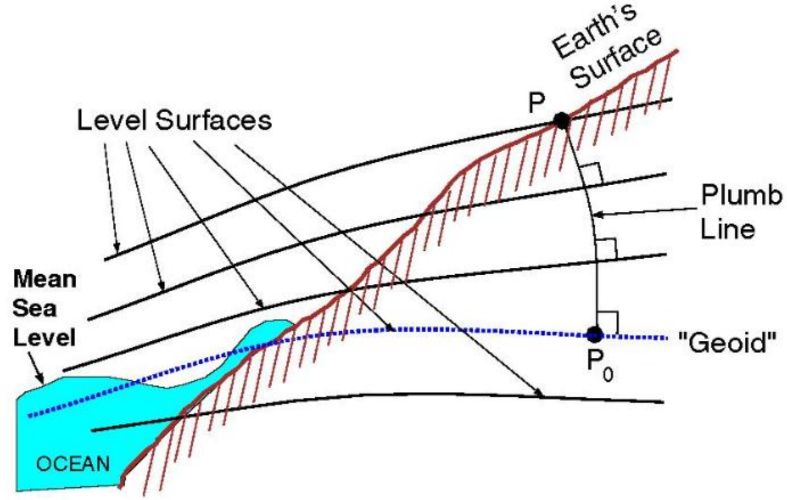


Thus the gravity vector **g** is perpendicular to the equipotential surface through the point.

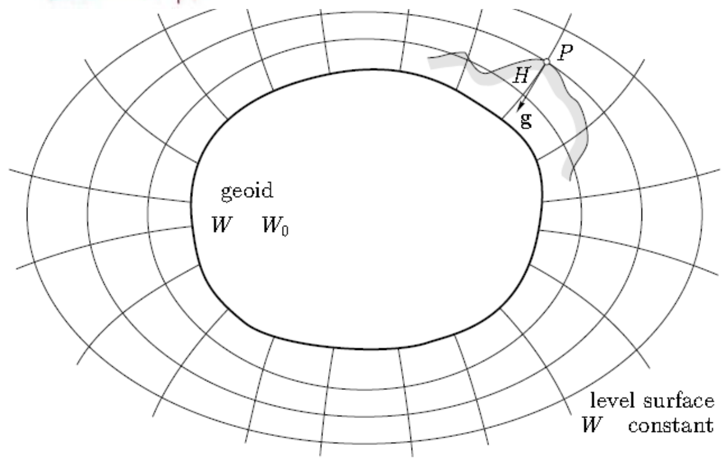
Plumb lines are lines to which **g** is tangent at every point.

They the perpendicular to the equipotential surfaces.

The **geoid** was first defined as the ”equipotential surface of the earth’s gravity field **coinciding with the mean sea level of the oceans**” by C.F. Gauss in 1828, and the name “geoid” was suggested by Listing in 1873



Due to variations of MSL both in space and time, this now taken to be “the equipotential surface which **best fits mean sea level at a certain epoch”**



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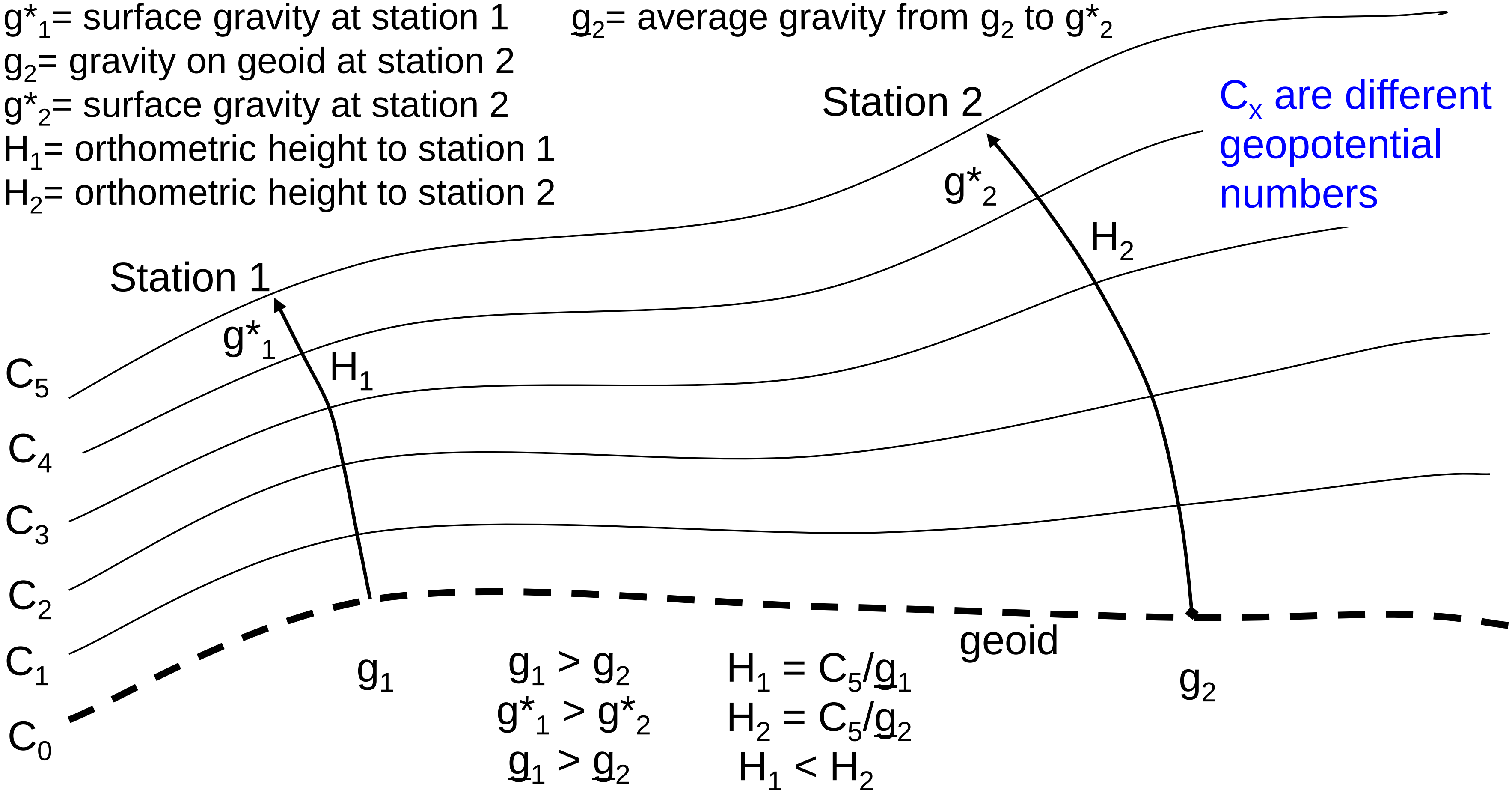
g_1 = gravity on geoid at station 1
g^*_1 = surface gravity at station 1
g_2 = gravity on geoid at station 2
g^*_2 = surface gravity at station 2
H_1 = orthometric height to station 1
H_2 = orthometric height to station 2

$\underline{g}_1$ = average gravity from g_1 to g^*_1
$\underline{g}_2$ = average gravity from g_2 to g^*_2



Note that surface location of station 1 is closer to the geoid than station 2

A steep gradient of geops (lines of equal geopotential) indicates higher gravity – less steep indicates lower gravity

The geops being farther apart beneath station 2 to reflect lower local mass and gravity

Hence, H_1 should be less than H_2 – even though both have the same geopotential

- The **normal gravity field** is an approximation of the Earth's potential field based on the Reference Ellipsoid
- It is required that the surface of this **level ellipsoid** be an equipotential surface (with same potential as the geoid $W_0 = U_0$)
- Defined by four parameters, e.g. **GRS80**:

Parameter and value	Description
$a = 6\,378\,137$ m	semimajor axis of the ellipsoid
$GM = 3\,986\,005 \cdot 10^8\ \mathrm{m^3\,s^{-2}}$	geocentric gravitational constant of the earth (including the atmosphere)
$J_2 = 108\,263 \cdot 10^{-8}$	dynamical form factor of the earth (excluding the permanent tidal deformation)
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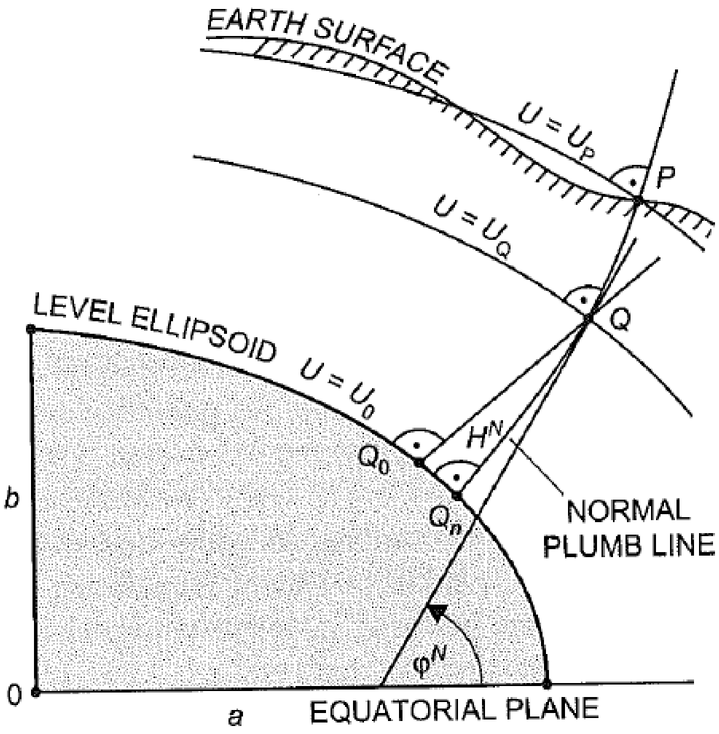

- **Normal gravity** γ:

$$g = \frac{a g_a \cos^2 f + b g_b \sin^2 f}{\sqrt{a \cos^2 f + b \sin^2 f}} \quad \text{(Somigliana 1929)}$$

$\gamma_a = 9.7803267715\,\mathrm{ms^{-2}}$ $\gamma_b = 9.8321863685\,\mathrm{ms^{-2}}$

Linking the "(Real)World" and "(Normal)Model" in Geodesy

- Approximation of the **shape**, **size** and **gravity field of the Earth** by **those of an equipotential ellipsoid of revolution** (*Reference Ellipsoid*) with the same mass and same rotational velocity as the Earth, i.e. GRS80
- Leads to approximations:

$$W = U + T \quad ; \quad g = \gamma + \delta g \quad ; \quad H = h - N$$

- U and its functionals can be computed using the RE parameters
- T is the disturbing potential
- Brun's equation: $N = \frac{T}{\gamma}$

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- Gravity disturbance:

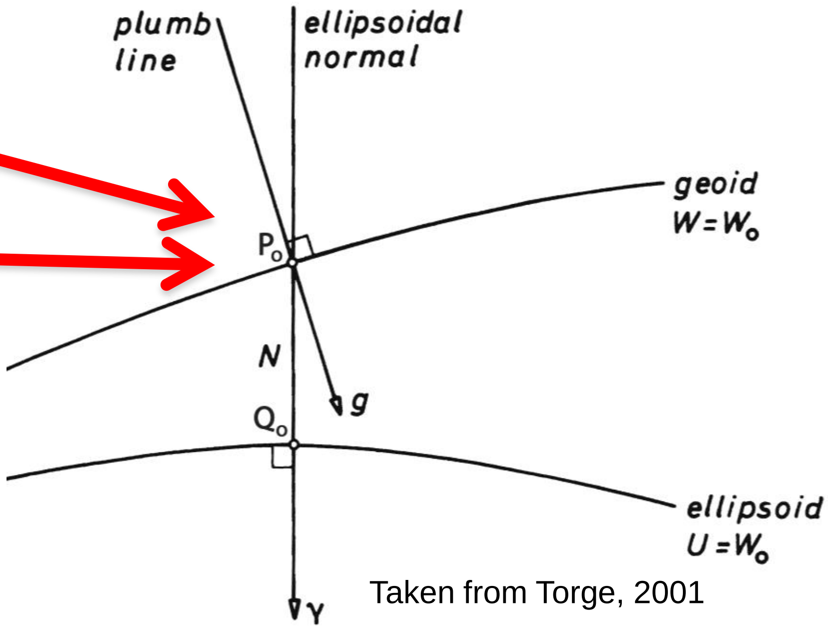
$$\delta g = g_{P_0} - \gamma_{P_0}$$

- Gravity anomaly:

$$\Delta g = g_{P_0} - \gamma_{Q_0}$$

- The fundamental equation of **physical geodesy**:

$$\Delta g = \delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}T = -\frac{\partial T}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}T$$



Taken from Torge, 2001

GBVP can be solved to compute geoid/quasi-geoid using gravity anomalies from all over the world... *see later slides*

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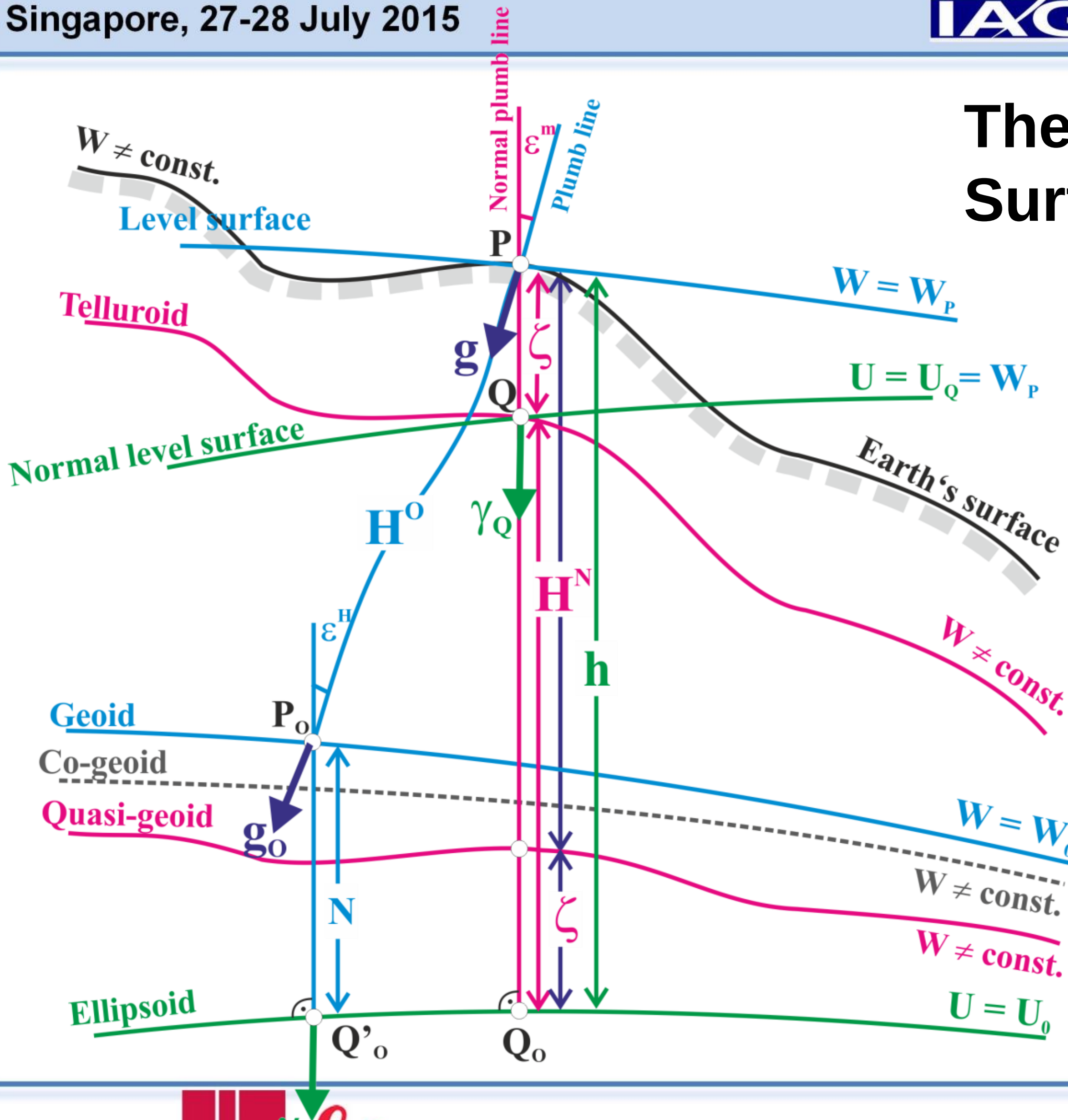
The Physical Height Surfaces & Quantities



- For orthometric heights**: the geoid**
- For normal heights: **the quasi-geoid**

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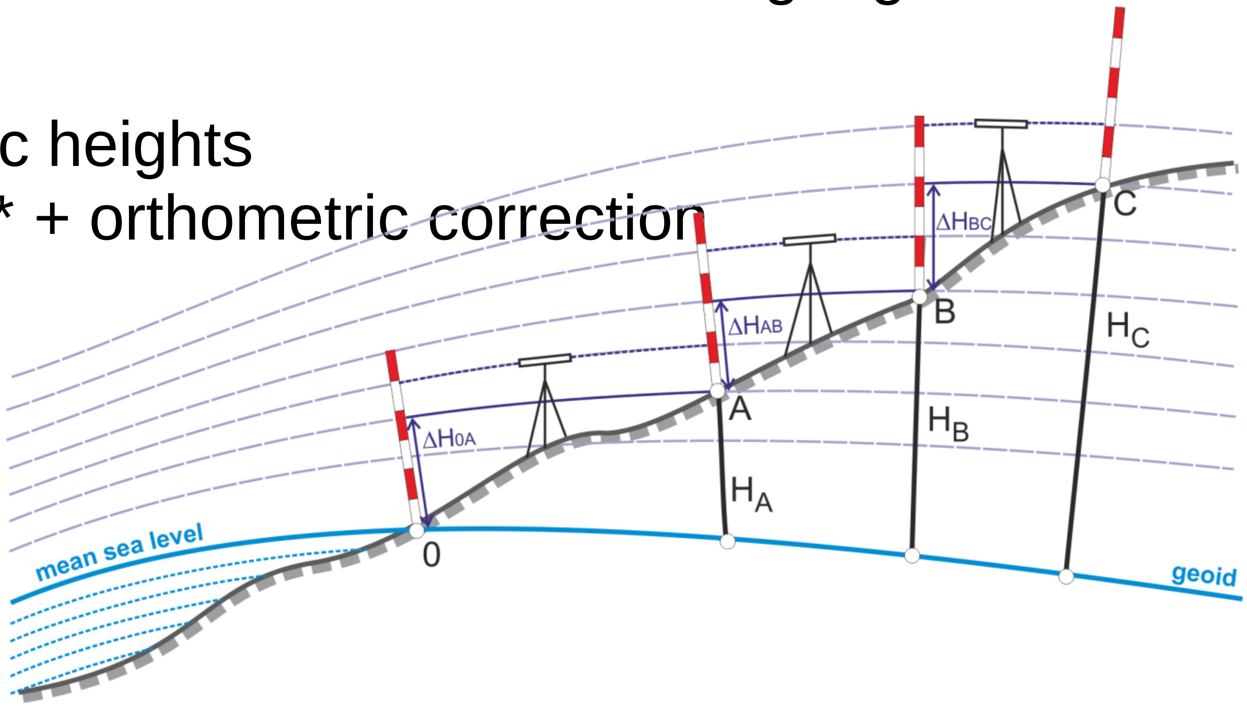
Spirit (Physical) Levelling

Defining the Vertical Reference System for Physical Heights

Reference surface:
Definition: the geoid
Realisation: MSL measured at one or more tide gauge

Vertical coordinate:
Definition: orthometric heights
Realisation: levelling* + orthometric correction



* Optical levelling =
spirit or trigonometrical

Challenges of Optical Levelling

- To reach the highest possible accuracy, it is necessary to minimise the influence of a number of **error sources...** ***See textbooks on Precise Levelling***
- Some of them cancel by using equal **backsights** and **foresights**
- It is especially important to minimise of **systematic effects**, since these otherwise tend to add up to large systematic errors over longer distances
- The most important **error sources** are:

Collimation error
Rod errors
Refraction
Vibration and swaying of the image
Vertical movements of the level and the rods (placed on temporary base plates)
Earth curvature
Magnetic field
Earth tides
Vertical crustal motion

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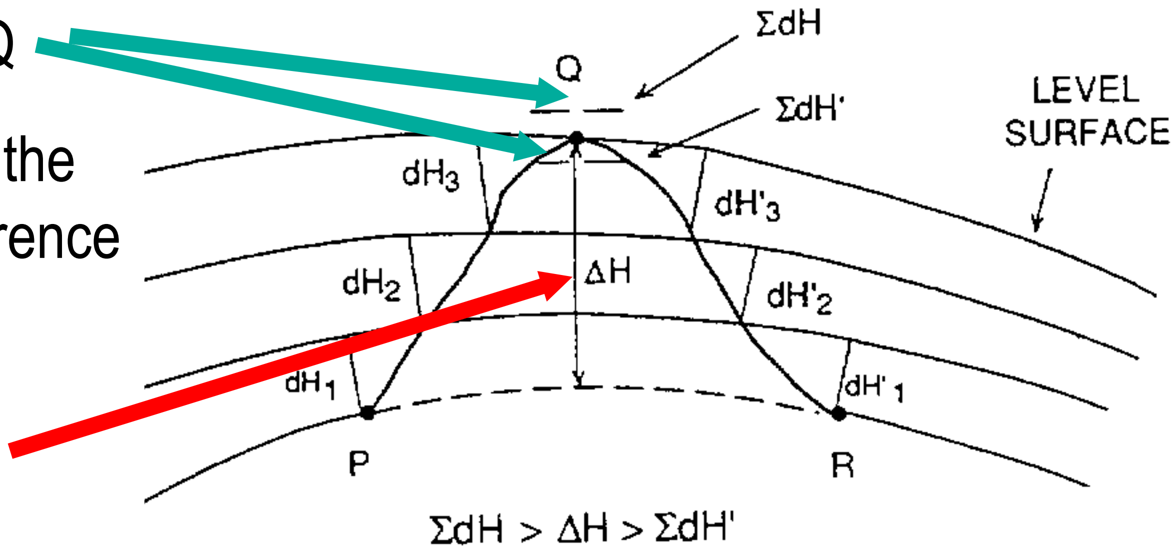
Levelled Heights & Orthometric Heights

- Sum of levelled height differences from R->Q is not equal to sum from P->Q
- Neither will be equal to the orthometric height difference ΔH, *due to the non-parallelism of the geopotential surfaces*



http://en.wikipedia.org/wiki/Orthometric_height

Physical Height Coordinate Types

- Since the potential depends only on position: $$\oint W = -\oint g\, dn = 0$$

- To avoid the path-dependence *convert* levelled height differences to **potential differences**:

$$dW = W_B - W_A = -\int_A^B g dn \approx -\sum_A^B g\delta n$$

- The *potential difference* between the point and the geoid can be converted back to physical heights (there are different kinds of physical heights)

- It is standard to avoid the minus sign by introducing the **geopotential number *C***:

$$C_A = W_0 - W_A \qquad \Rightarrow \qquad dC = -dW = g \cdot dn$$

$$\Delta C_{AB} = C_B - C_A = \int_A^B g \cdot dn \approx \sum_{i=1}^{n} g_i \delta n_i$$

Unit: 1 gpu = 10 m^2/s^2 = 1 kGalm (approximately the same magnitude as height, about 2% difference)

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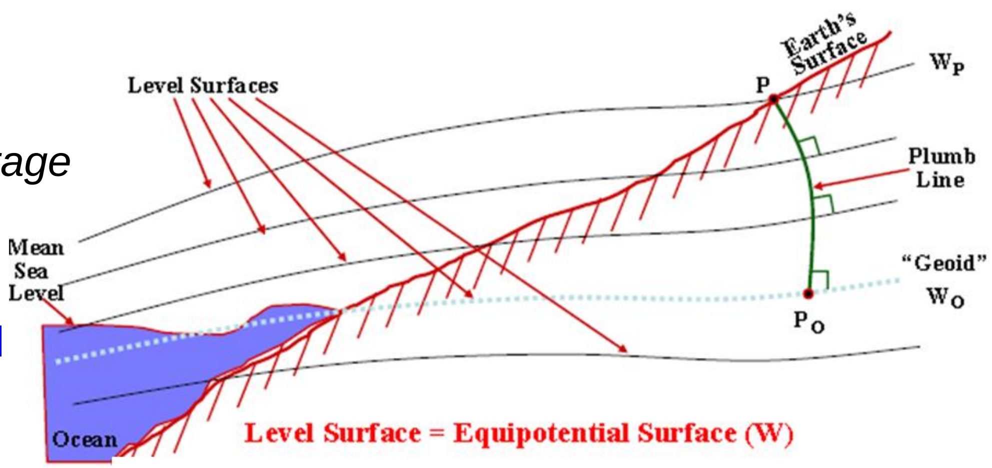
Orthometric Heights

- OH can be computed from the **geopotential number**:

$$C_P = W_0 - W_P = \int_0^{H_P} g \cdot dn = H_P \frac{1}{H_P} \int_0^{H_P} g \cdot dn = H_P \cdot \bar{g}_P \qquad H_P = \frac{C_P}{\bar{g}_P}$$

$\bar{g}$ is the average value of gravity along the plumb line

- Computation of OH in theory requires that **gravity is known inside the masses**, along the plumb line
- *In practice, one can approximate the average gravity using some sort of model from surface gravity measurements*
- **Due to non-parallelism of equipotential surfaces, points with same OH are not on the same equipotential surface**



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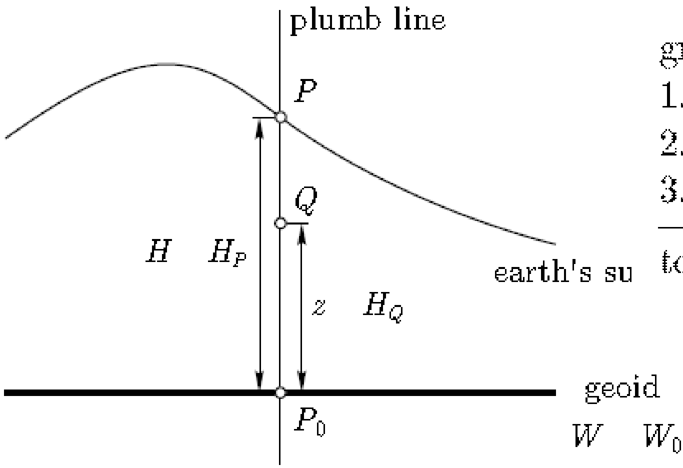
Helmert (Orthometric) Heights

Mean gravity along the plumb line is computed by means of a **Prey reduction** with standard density $\rho_0 = 2.67$



gravity measured at P	g_P
1. remove Bouguer plate	$-0.1119\,(H_P - H_Q)$
2. free-air reduction from P to Q	$+0.3086\,(H_P - H_Q)$
3. restore Bouguer plate	$-0.1119\,(H_P - H_Q)$
together: gravity at Q	$g_Q = g_P + 0.0848\,(H_P - H_Q)$.

$$H = \frac{C}{\bar{g}} \qquad \bar{g} = \frac{1}{H}\int_0^H g(z)\,dz = \frac{1}{H}\int_0^H \left(g + 0.0848(H - z)\right)dz = g + \frac{0.0848}{H}\left[Hz - \frac{z^2}{2}\right]_0^H = g + 0.0424 \cdot H$$

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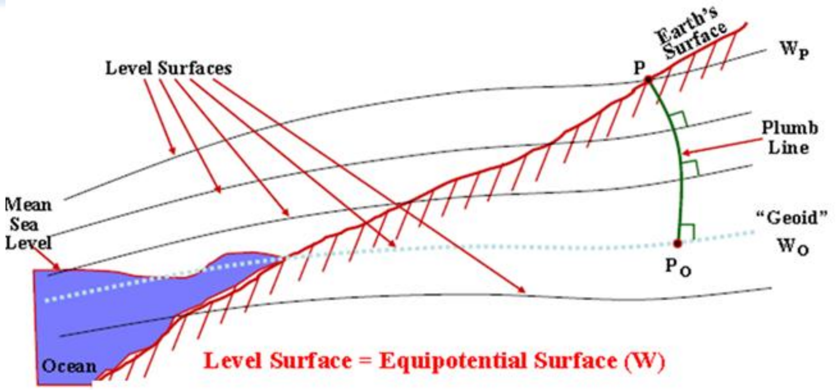
Dynamic Heights

$$H_P^{dyn} = \frac{C_P}{\gamma_0}$$



$$\gamma_0 = \text{ arbitrary constant } = \gamma_0^{45^\circ}$$

- Simple conversion of the geopotential number to height units
- Points along an equipotential surface have same dynamic height (DH), i.e. undisturbed water surfaces have same DH
- But **no geometrical meaning**
- Large dynamic corrections are obtained in many areas of the Earth
- *Primarily used before geopotential numbers were adapted by IAG in 1955*
- *Used in oceanography and meteorology*

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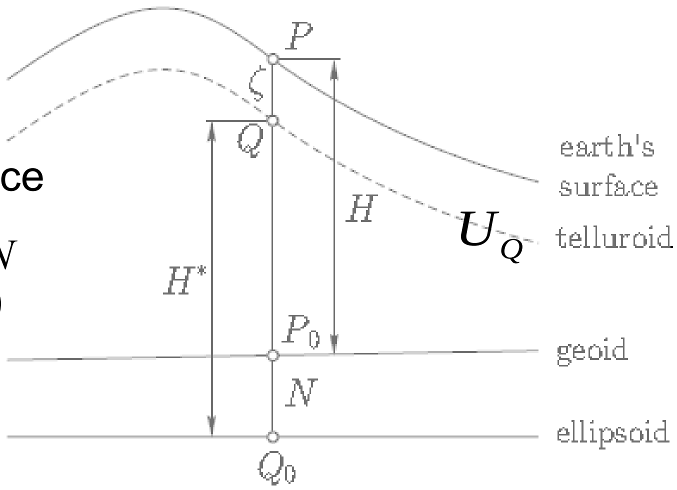
Normal Heights

- The geometric distance between the **ellipsoid** and **telluroid** along the normal plumb line:

$$C_P = W_0 - W_P = U_0 - U_Q = \int_0^{H_P^N} \gamma \cdot dn = H_P^N \frac{1}{H_P^N} \int_0^{H_P^N} \gamma \cdot dn = H_P^N \cdot \bar{\gamma}_P$$

$$H_P^N = \frac{C_P}{\bar{\gamma}_P} \qquad \bar{\gamma}_P = \gamma_0 \left(1 - \frac{1}{a} \left(1 + f + m - 2f \sin^2 \phi \right) H^N + \ldots \right)$$

- The **telluroid** is defined as the surface at which the normal potential U_Q is equal to real potential W_P at the Earth's surface
- The **height anomaly** ζ is the analogue of the geoid height N in Molodenskii's Boundary Value Problem *(see other slides)*
- **The NH and height anomaly can be estimated without knowledge of the topographic density**



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Normal Orthometric Heights

- Except for the unknown topographic density, the only approximation is the discretisation of the levelling process:

$$\Delta C_{AB} = \int_A^B g \cdot dn \approx \sum_{i=1}^{n} g_i \delta n_i$$

- If gravity is unknown (not measured) best guess would be to use **normal gravity** instead:

$$\Delta C_{AB} = \int_A^B \gamma(\phi, h) \cdot dn \approx \sum_{i=1}^{n} g_i(\phi_i, h_i) \delta n_i$$

- Applied mostly for orthometric heights, leading to **normal orthometric heights**
- *Do not confuse **normal orthometric heights** with **normal heights**!!*
- *Many older vertical datums utilise normal orthometric heights, e.g. AHD71*

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Corrections to Levelled Height Differences

- Dynamic correction:

$$\Delta H_{AB}^{dyn} = \Delta n_{AB} + DC_{AB}; \qquad \mathrm{DC}_{AB} = \int_A^B \frac{g-\gamma_o^{45}}{\gamma_o^{45}} dn \approx \sum_A^B \frac{g-\gamma_o^{45}}{\gamma_o^{45}} \delta n$$

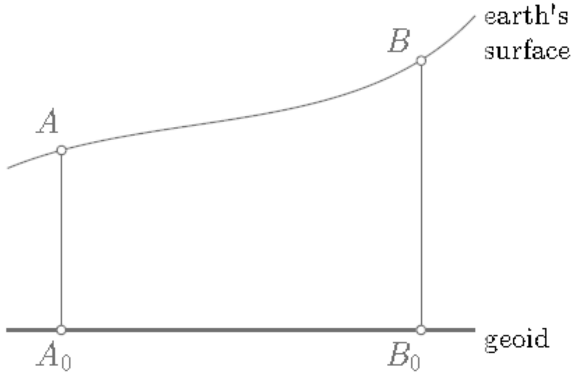
- Orthometric correction:

$$\Delta H_{AB} = \Delta n_{AB} + OC_{AB}$$

$$\mathrm{OC}_{AB} = \int_A^B \frac{g-\bar{g}}{\bar{g}} dn \approx \sum_A^B \frac{g-\bar{g}}{\bar{g}} \delta n$$

$$\Delta H_{AB} = H_B - H_A = \Delta H_{A,B}^{dyn} + \left(H_A^{dyn} - H_A\right) - \left(H_B^{dyn} - H_B\right)$$

$$-\sum_A^B \frac{g-\gamma_0^{45}}{\gamma_0^{45}} \delta n + \frac{\bar{g}_A - \gamma_0^{45}}{\gamma_0^{45}} H_A - \frac{\bar{g}_B - \gamma_0^{45}}{\gamma_0^{45}} H_B = DC_{AB} + DC_{A_0A} - DC_{B_0B}$$



- Normal correction: $\Delta H_{AB} = \Delta n_{AB} + NC_{AB};$

$$\mathrm{NC}_{AB} = \int_A^B \frac{g-\bar{\gamma}}{\bar{\gamma}} dn \approx \sum_A^B \frac{g-\bar{\gamma}}{\bar{\gamma}} \delta n$$

Convert levelled height increments to orthometric height increments:

$$\Delta H_{AB} = \sum dH_{AB} + OC_{AB}$$

Where the "orthometric correction" (OC) is:

$$OC_{AB} = \sum_{A}^{B} \frac{g - \gamma_r}{\gamma_r} . dH + \frac{\bar{g}_A - \gamma_r}{\gamma_r} . H_A - \frac{\bar{g}_B - \gamma_r}{\gamma_r} . H_B$$

Where $\bar{g}_A$ and $\bar{g}_B$ are mean values of gravity along plumbline at A & B, γ_r is normal gravity at "reference latitude" (e.g. 45°), and g is surface gravity at measured height difference.

Hence gravity should be measured together with levelled height!

$\bar{g}_A$ and $\bar{g}_B$ can be replaced by g_A and g_B

For **normal orthometric height** increments: $\bar{g}_A$ and $\bar{g}_B$ can be replaced by $\bar{\gamma}_A$ and $\bar{\gamma}_B$

Sea Level Reference Surfaces

- **Sea surface height (SSH)**: vertical distance of sea surface wrt ellipsoid (geometric height from satellite altimetry of GNSS):

$$SSH = h_s - r_j$$
$$SSH = N + DT$$

- **Dynamic topography (DT)**: difference between sea surface and geoid (physical height):

$$DT = h_s - r_j - N$$

Mainly caused by tides, currents, winds, Earth rotation, seasonal effects, temperature, salinity, etc.
Determined using ocean (dynamic) models based on hydrostatic equilibrium laws.

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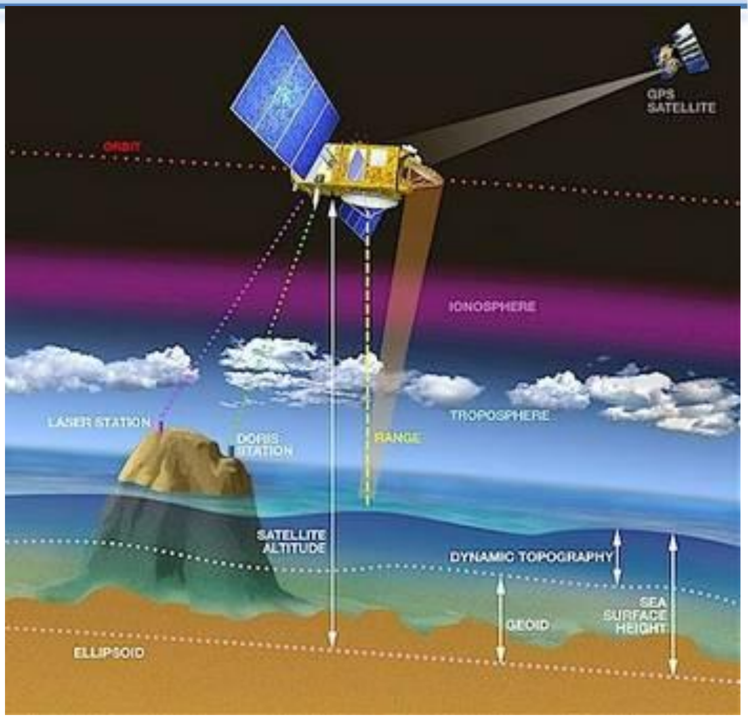
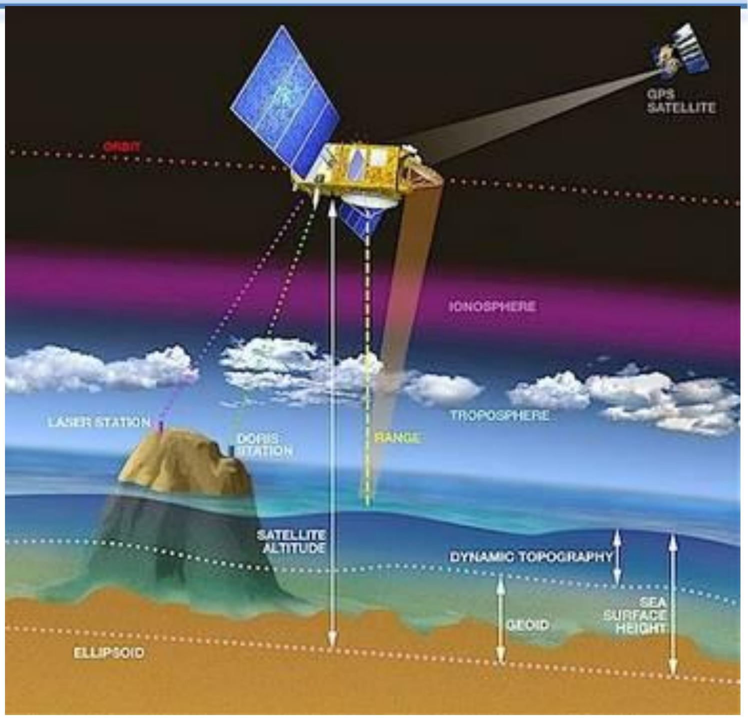
Source: http://www.aviso.oceanobs.com



- **Mean sea surface (MSS)** : long-term average of sea surface heights:

$$MSS = \frac{1}{y}\sum_{y} MSH$$

Source: http://www.aviso.oceanobs.com



- DT is separated in **mean dynamic topography MDT** (considered semi-stationary) and **dynamic ocean topography DOT** (time-variable part of DT):

$$DT = MDT + DOT$$

- The **MDT** is the "oceanic relief", mainly caused by geostrophic currents. Also referred to as **Sea Surface Topography** (**SSTop**):

$$MDT = SSTop = MSS - N$$

- **DOT** contains contributions from wind and other high frequency effects. Usually inter-annual, or other short-term, variations from MDT. Also referred to as **Sea Level Anomalies (SLA)**

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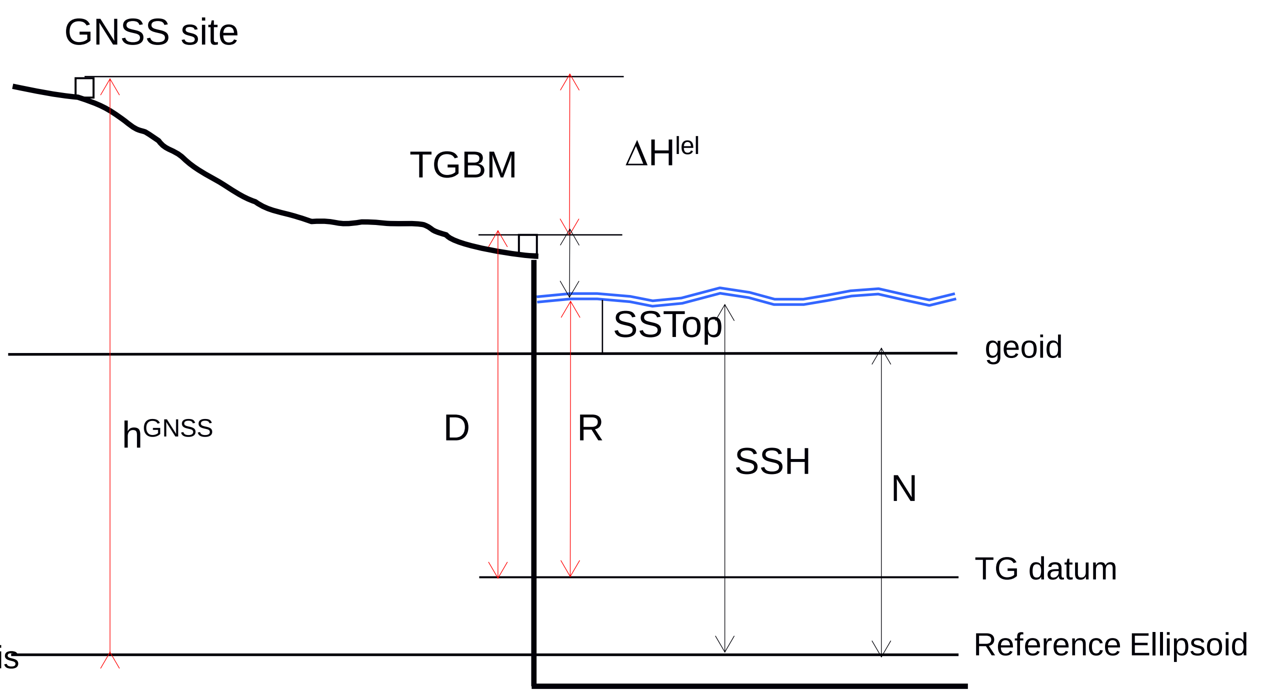
$$SSH = h_s = N + SSTop = N + H_s$$
$$= h^{GNSS} - DH^{lev} - D - R$$
$$SSTop = H_s = SSH - N = h_s - N$$
$$= h^{GNSS} - DH^{lev} - D - R - N$$

Easy to neglect a term... and to assume static quantities... will vary spatially & temporally

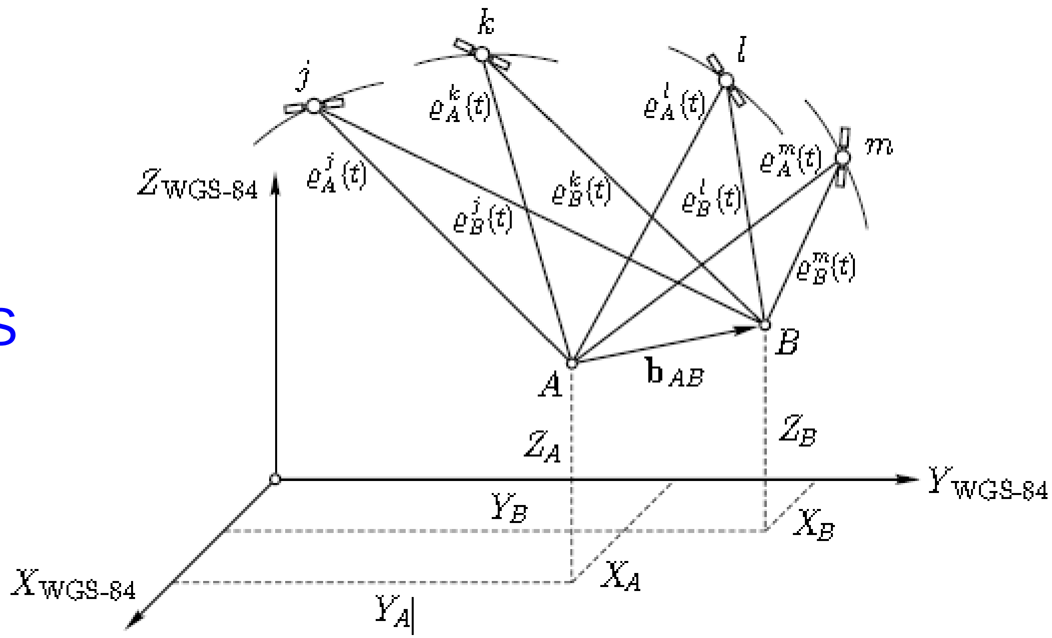


Courtesy: M. Sideris

GNSS (Geometric) Levelling

- GNSS gives **geometric (ellipsoidal)** heights independently of the gravity field
- Initially as 3-dimensional coordinates in geocentric Cartesian frame, e.g. ITRF
- Converted to geodetic coordinate quantities that refer to the Reference Ellipsoid

Relative or Absolute ellipsoidal heights, depending upon GNSS positioning technique used...



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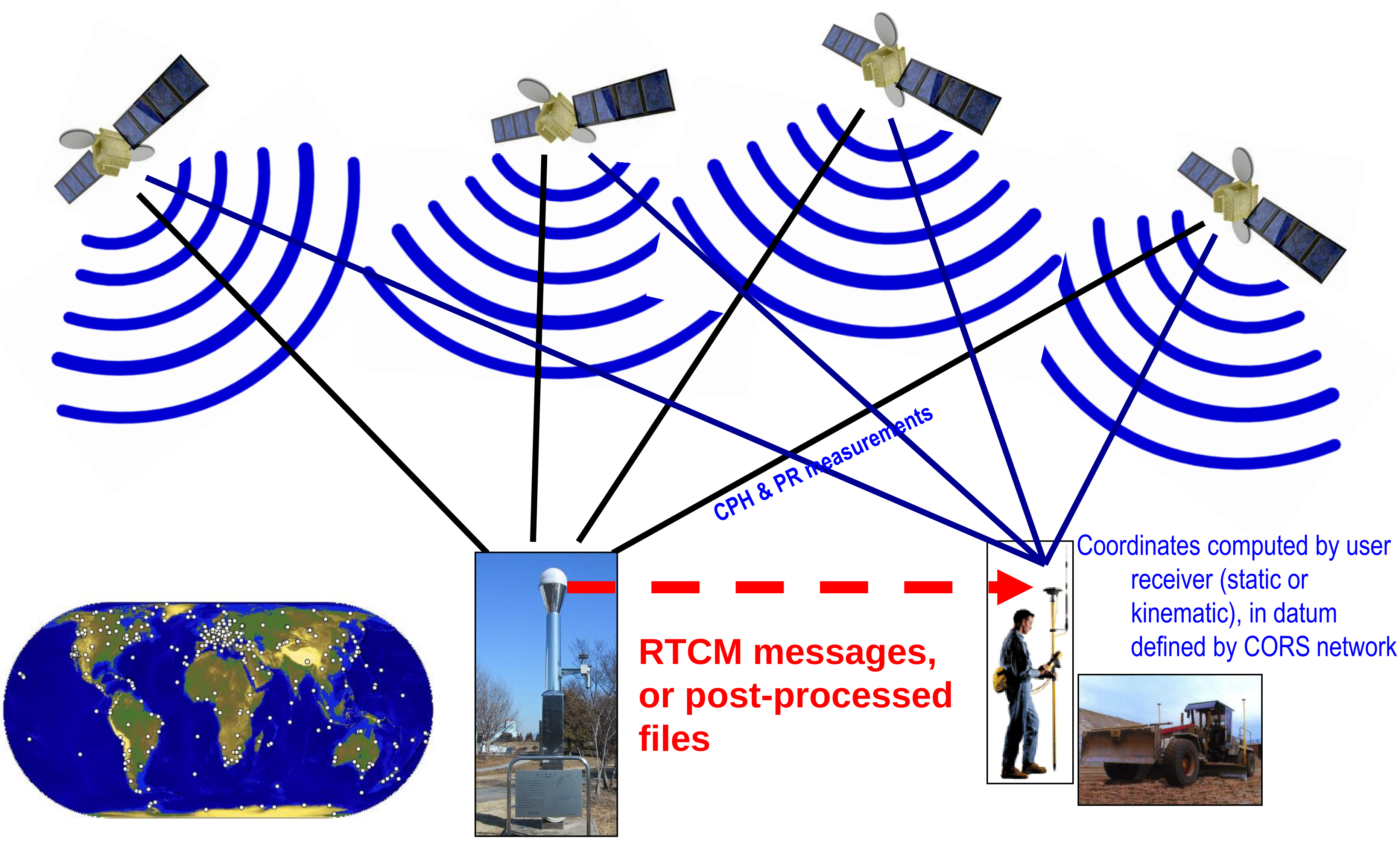
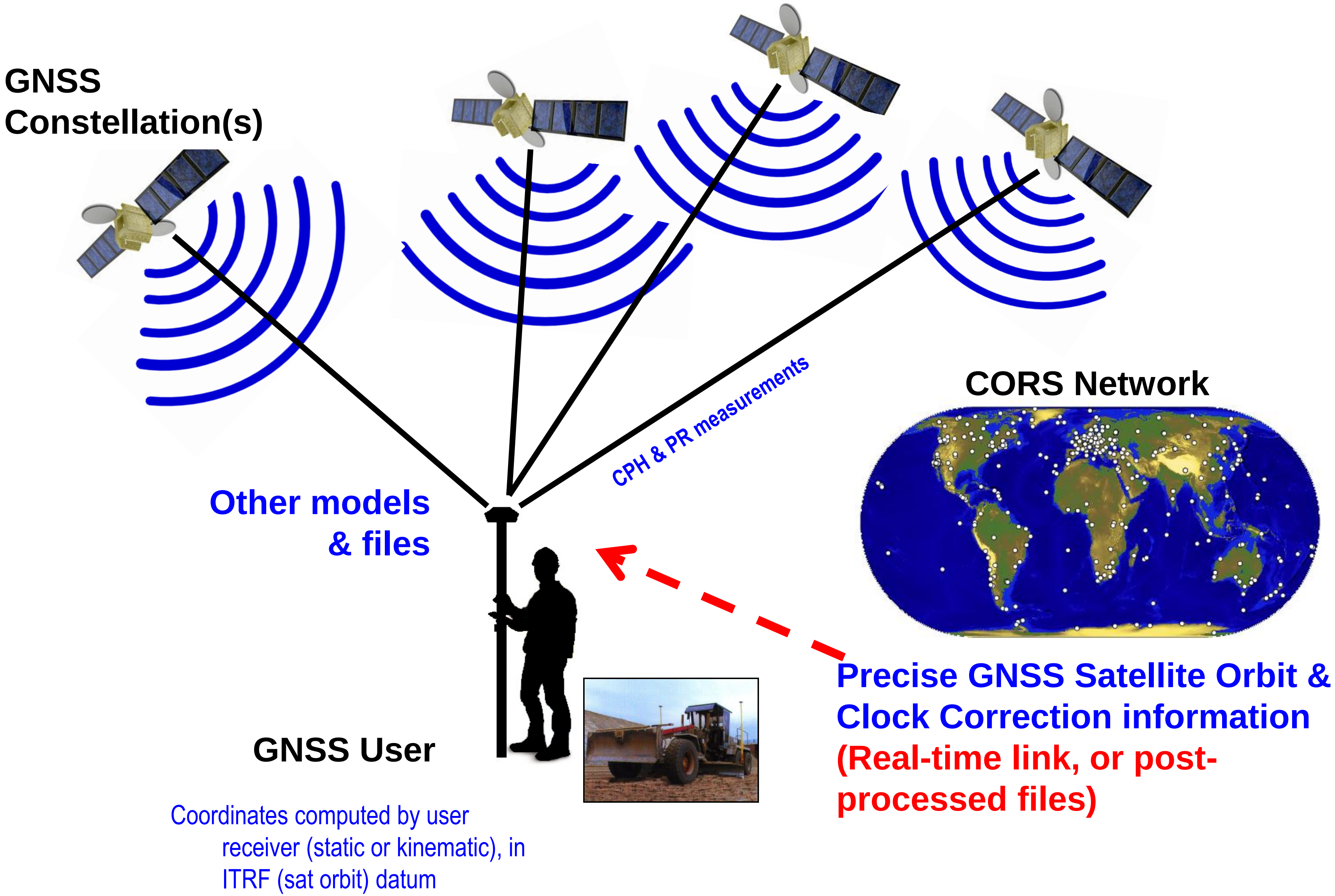
Differential GNSS (DGNSS)



Precise Point Positioning (PPP)

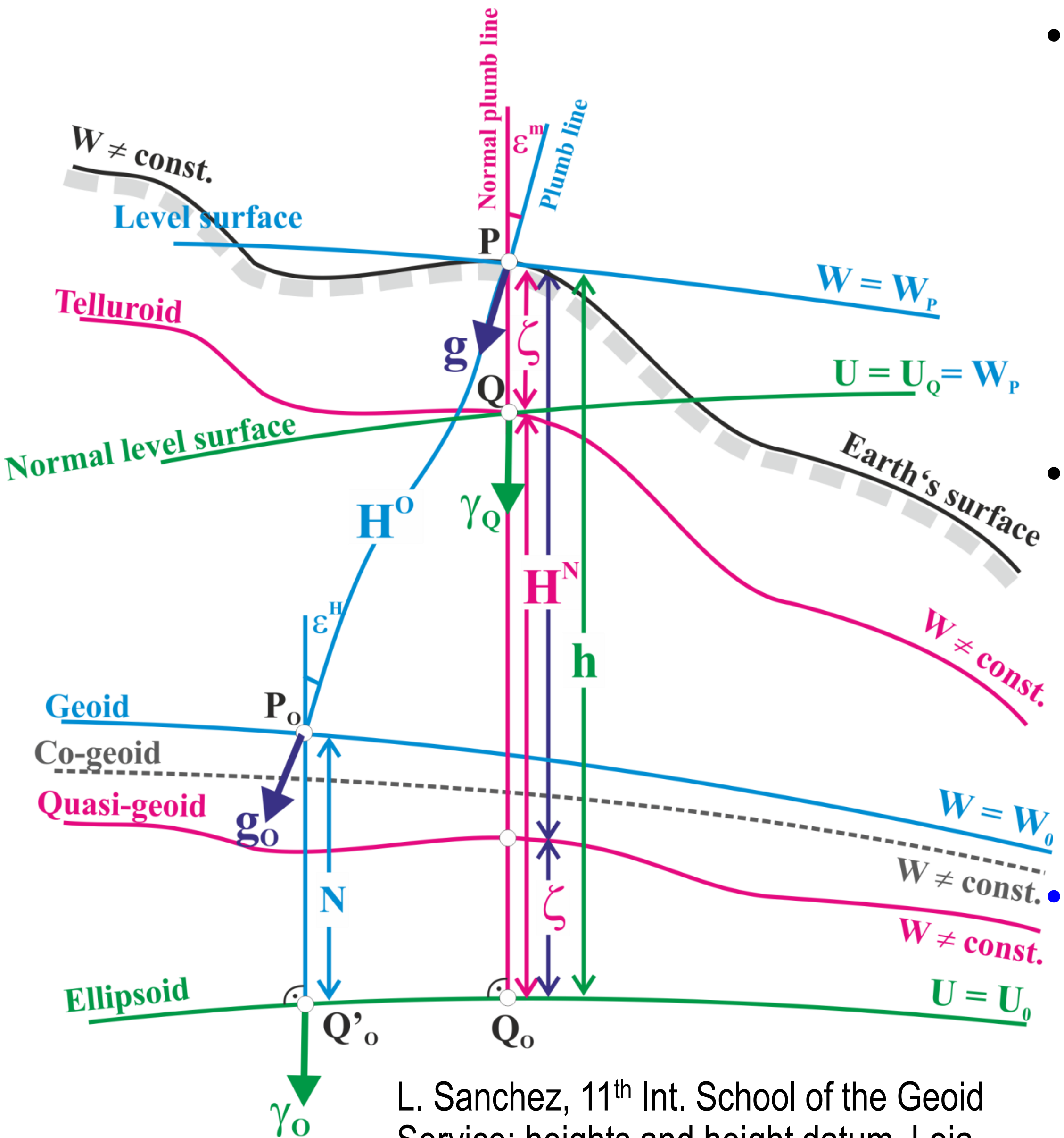


Why GNSS?

- Replaces spirit or trigonometrical levelling as they are: Costly; Laborious; Time-consuming
- Today it is possible to determine height differences using precise GNSS positioning techniques with:
  - ~5-10mm uncertainty over long to very long distances using **static** method
  - ~10-30mm uncertainty in **real time** using single station RTK or Network-RTK
- For many applications this is sufficient, assuming a *consistent* high quality geoid model is available... But challenge is consistency with legacy height data derived from optical levelling
- *Over short distances spirit levelling is still the most accurate height difference determination technique*

Some Definitions: Physical Height Systems

Physical Heights: Recap



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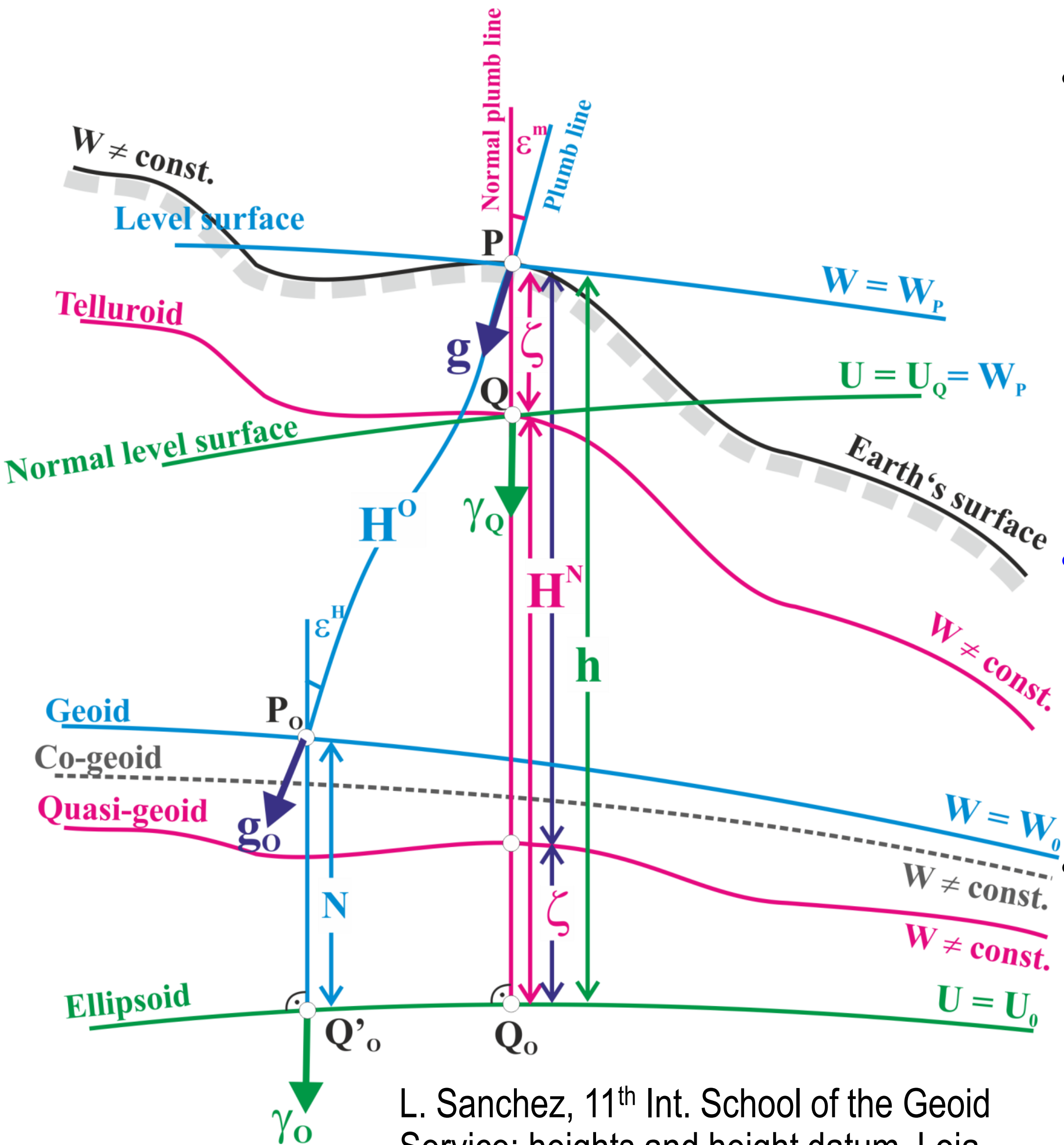
- The **telluroid** is the surface defined by those points Q, whose normal potential U_Q is identical with the actual potential W_P of the points *P* on the Earth's surface, i.e. $U_Q = W_P$
- The **co-geoid** is the *estimated* geoid – the real geoid cannot be known, because the actual mass distribution and actual gravity vertical gradient are not known precisely
- Assumption of hypotheses concerning gravity produces "something similar" to the geoid (i.e. the co-geoid), *but not the true geoid*

Physical Heights: Recap



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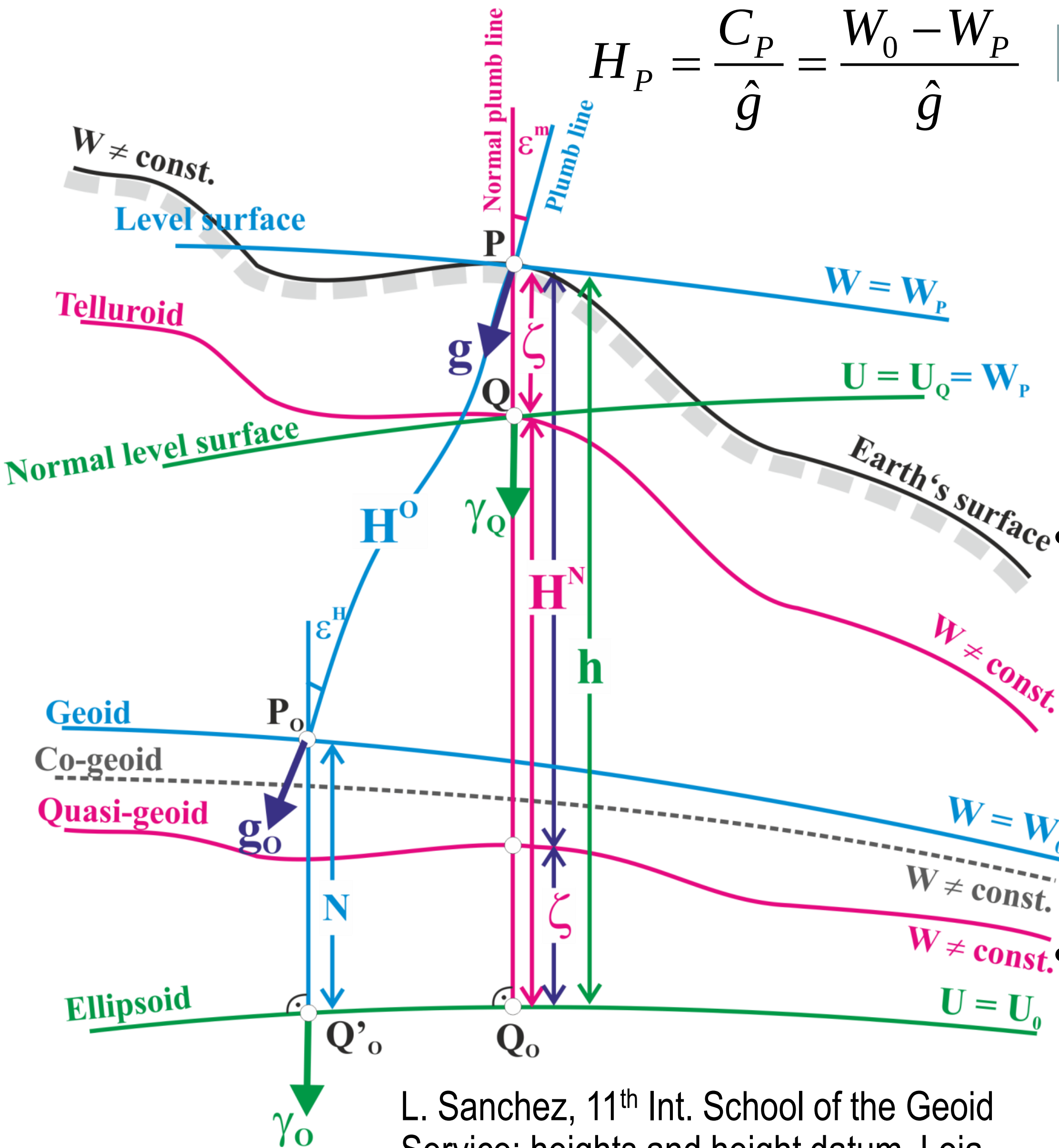
- The **height anomaly** ζ is the distance, along the normal plumb line, between the Earth's surface and the telluroid. *When plotted above the ellipsoid the resulting surface is called the* ***quasi-geoid***
- Telluroid, quasi-geoid and co-geoid *are not equipotential surfaces* (the gravity vector is not perpendicular to them)
- Geoid and quasi-geoid are identical in ocean areas

Physical Heights: Recap



$$H_P = \frac{C_P}{\hat{g}} = \frac{W_0 - W_P}{\hat{g}}$$

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- If $\hat{g}$ is the **mean real gravity value**, along the plumb line, between Earth's surface and geoid, we get **orthometric heights** (if instead use **normal gravity**, then we get **normal orthometric heights**)
- If $\hat{g}$ is **the mean normal gravity value** between telluroid and ellipsoid (or between Earth's surface and the quasi-geoid), we get **normal heights**
- If $\hat{g}$ is a **constant normal gravity value**, we get **dynamic heights**

- **Geopotential numbers** themselves are a very important result of levelling. The potential difference is directly related to how water flows. *They have, however, no geometrical meaning.*
- **Dynamic heights** also have no geometrical meaning.
- **Orthometric heights** are natural heights above zero reference surface. Main problem with OHs is they depend on a density hypothesis for correction of surface observed gravity. Various approximations have been used, e.g. **Helmert orthometric heights**.
- **Normal heights** do not have an obvious physical meaning. The advantage is that they can be computed completely without any density hypothesis.
- **Normal orthometric heights** commonly used because no observed gravity values are needed for OC reduction of levelled height differences.

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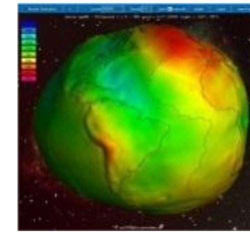
Geoid Models & Computations

Geoid Models (1):

Global Gravity Model

- Derived from analysis of satellite orbit perturbations or from satellite-satellite tracking (GRACE) or gradiometry (GOCE) measurements
- Long wavelength (not as detailed as regional or local models)
- Consist of Spherical Harmonic Coefficients (SHM) & RE parameters
- Modern GGMs to very high degree & order
- Geoid, gravity & geopotential quantities can be computed from SHM
- Some include surface gravity data, e.g. EGM2008
- Most are purely satellite-based, including time-varying (see ICGEM web site)

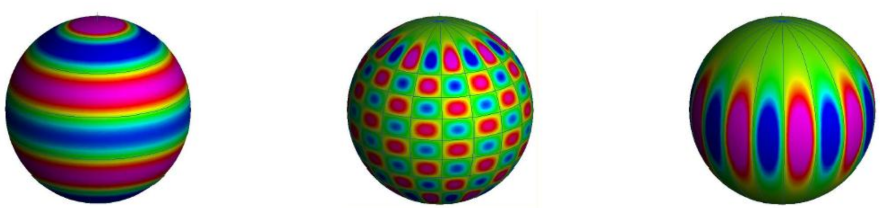
T is the disturbing potential (geopotential W minus normal potential U):



$$T(r,\theta,\lambda)=\frac{GM}{r}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n}\bar{P}_{nm}(\cos\theta)\left(\bar{C}_{nm}\cos m\lambda+\bar{S}_{nm}\sin m\lambda\right)$$

(r,θ,λ)	spherical coordinates [radial distance from geocentre, co-latitude, longitude]
a	ellipsoid's semi-major axis
M	Earth's mass
G	gravitational constant
$\bar{P}_{nm}$	fully normalised spherical harmonic functions of degree *n* and order *m*
$\bar{C}_{nm}, \bar{S}_{nm}$	fully normalised spherical harmonic coefficients (SHM)

Brun's equation gives geoid height from SHM: $N=\dfrac{T}{\gamma}$



zonal: $\ell=6,\ m=0$ tesseral: $\ell=16,\ m=9$ sectorial: $\ell=9,\ m=9$

Figure 4: Examples for spherical harmonics $P_{\ell m}(\sin\varphi)\cdot\cos m\lambda$ [from -1 (blue) to $+1$ (violet)]

ICGEM International Center for Global Gravity Field Models

icgem.gfz-potsdam.de/ICGEM/

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ICGEM

- GFZ Potsdam
- ICGEM Home
- Table of Models
- Evaluation of Models
- References
- Gravity Visualization
- Calculation Service
- Theory
- Models from Dedicated Time Periods
- Models related to Topography
- Animation of Monthly Models
- Visualization of Spherical Harmonics
- Latest Changes
- Guest Book

Other Celestial Bodies (Moon, Venus, Mars)

- Table of Models
- Gravity Visualization
- Calculation Service

IAG

GFZ Helmholtz Centre Potsdam

I C G E M

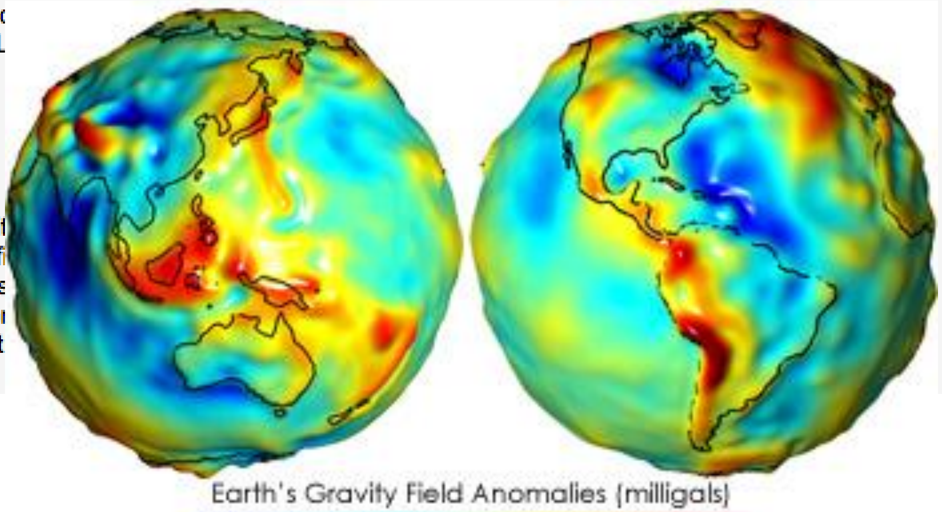
International Centre for Global Earth Models (ICGEM)

ICGEM is one of six centres of the International Gravity Field Service (IGFS) of the International Association of Geodesy (IAG). The other five Centres are

- Bureau Gravimetrique International (BGI) at CNES / CRGS, Toulouse, France
- Digital Elevation Model Centre (DEM) at Montfort University, UK
- International Centre for Earth Tides (ICET) at Obs. Royal de Belgique, Brussels, Belgium
- International Geoid Service (IGeS) at Politecnico d
- Technical Support Centre of IGFS at NGA, Saint L

Services of ICGEM

- collecting and archiving of all existing global gravit
- web interface for getting access to global gravity fi
- web based visualization of the gravity field models
- web based service for calculating different functio
- web site for tutorials on spherical harmonics and t



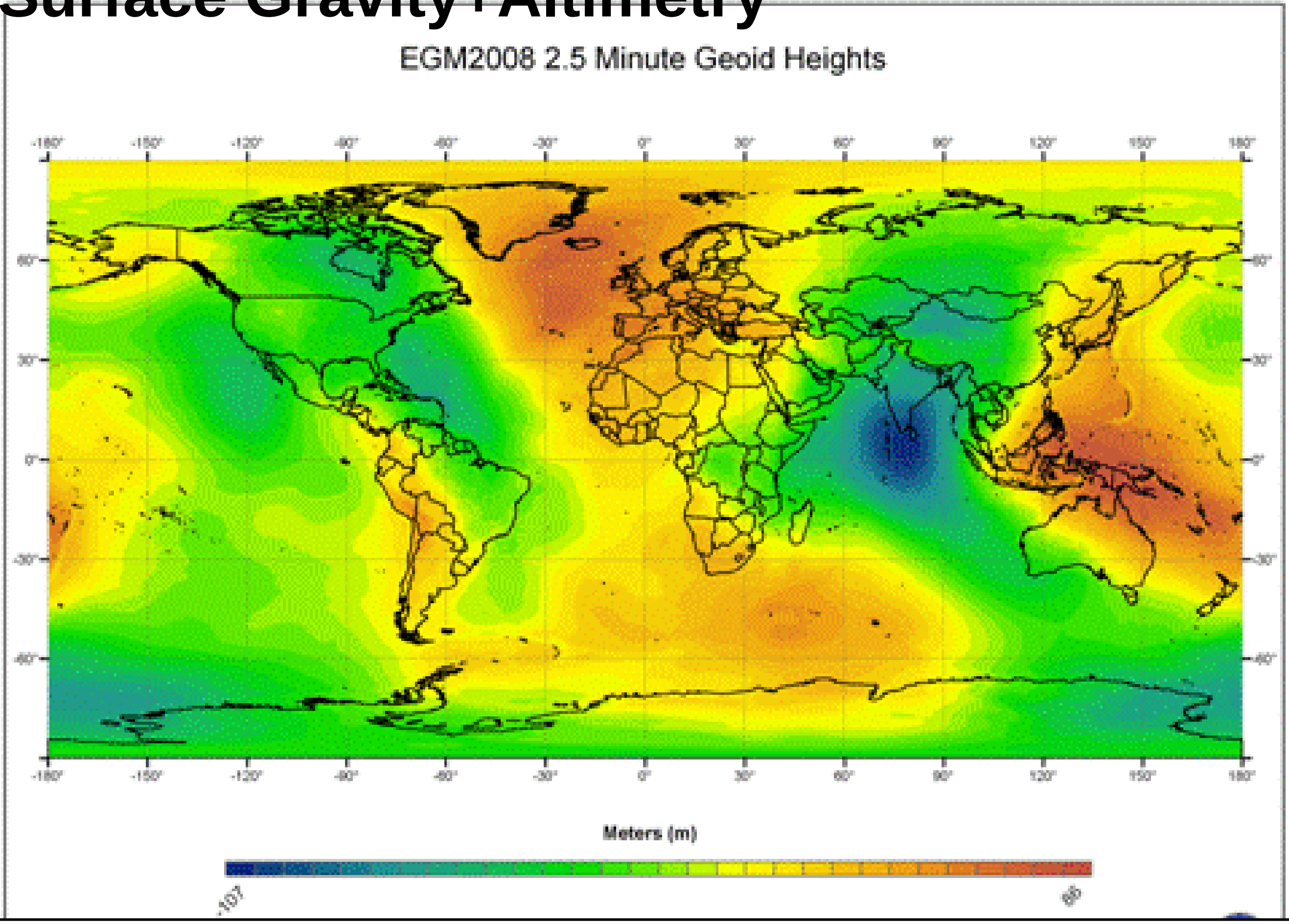
Nr ▲	Model ⇅	Year ⇅	Degree ⇅	Data ⇅	Reference ⇅
151	GGM05G	2015	240	S(Grace,Goce)	Bettadpur et al, 2015
150	GOCO05s	2015	280	S(see model)	Mayer-Gürr, et al. 2015
149	GO_CONS_GCF_2_SPW_R4	2014	280	S(Goce)	Gatti et al, 2014
148	EIGEN-6C4	2014	2190	S(Goce,Grace,Lageos),G,A	Förste et al, 2014
147	ITSG-Grace2014s	2014	200	S(Grace)	Mayer-Gürr et al, 2014
146	ITSG-Grace2014k	2014	200	S(Grace)	Mayer-Gürr et al, 2014
145	GO_CONS_GCF_2_TIM_R5	2014	280	S(Goce)	Brockmann et al, 2014
144	GO_CONS_GCF_2_DIR_R5	2014	300	S(Goce,Grace,Lageos)	Bruinsma et al, 2013
143	JYY_GOCE04S	2014	230	S(Goce)	Yi et al, 2013
142	GOGRA04S	2014	230	S(Goce,Grace)	Yi et al, 2013
141	EIGEN-6S2	2014	260	S(Goce,Grace,Lageos)	Rudenko et al. 2014
140	GGM05S	2014	180	S(Grace)	Tapley et al, 2013
139	EIGEN-6C3stat	2014	1949	S(Goce,Grace,Lageos),G,A	Förste et al, 2012
138	Tongji-GRACE01	2013	160	S(Grace)	Shen et al, 2013
137	JYY_GOCE02S	2013	230	S(Goce)	Yi et al, 2013
136	GOGRA02S	2013	230	S(Goce,Grace)	Yi et al, 2013
135	ULux_CHAMP2013s	2013	120	S(Champ)	Weigelt et al, 2013
134	ITG-Goce02	2013	240	S(Goce)	Schall et al, 2014
133	GO_CONS_GCF_2_TIM_R4	2013	250	S(Goce)	Pail et al, 2011
132	GO_CONS_GCF_2_DIR_R4	2013	260	S(Goce,Grace,Lageos)	Bruinsma et al, 2013
131	EIGEN-6C2	2012	1949	S(Goce,Grace,Lageos),G,A	Förste et al, 2012
130	DGM-1S	2012	250	S(Goce,Grace)	Farahani, et al. 2013
129	GOCO03S	2012	250	S(Goce,Grace,...)	Mayer-Gürr, et al. 2012

Geoid Models (2):

Regional Gravity Models (also known as **"gravimetric" geoids**)

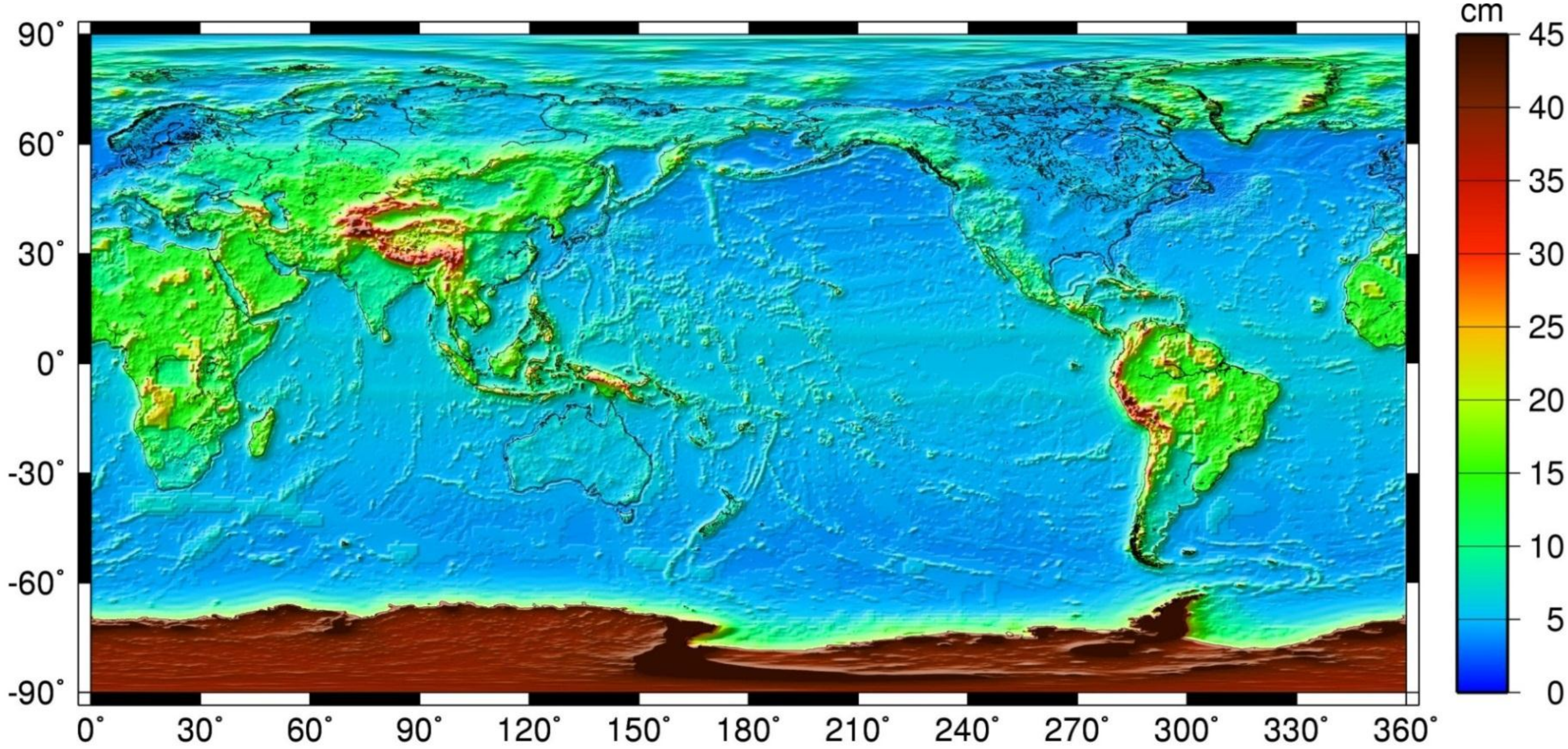
- General approach is to start with a GGM, and refine/compute using "Remove-Compute-Restore" Technique
- Gravity observations from multiple sources: terrestrial, shipborne, airborne, & altimetric
- Adopt a computational approach: Stokes vs. Molodenskii... *See later slides*
- Various computational techniques: LS collocation, FFT, numerical integration, etc.
- Must account for terrain (& other corrections to gravity anomalies)

Satellite+Surface Gravity+Altimetry

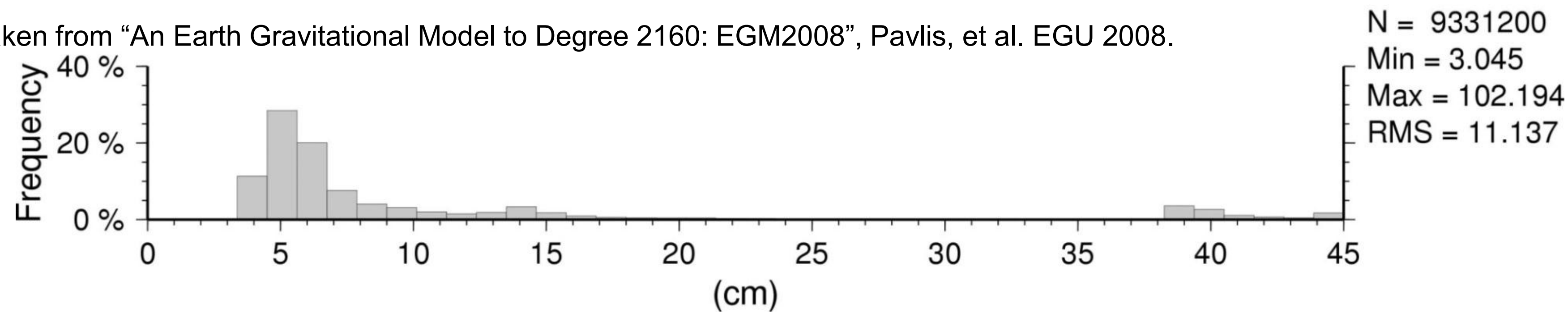


http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008/egm08_wgs84.html

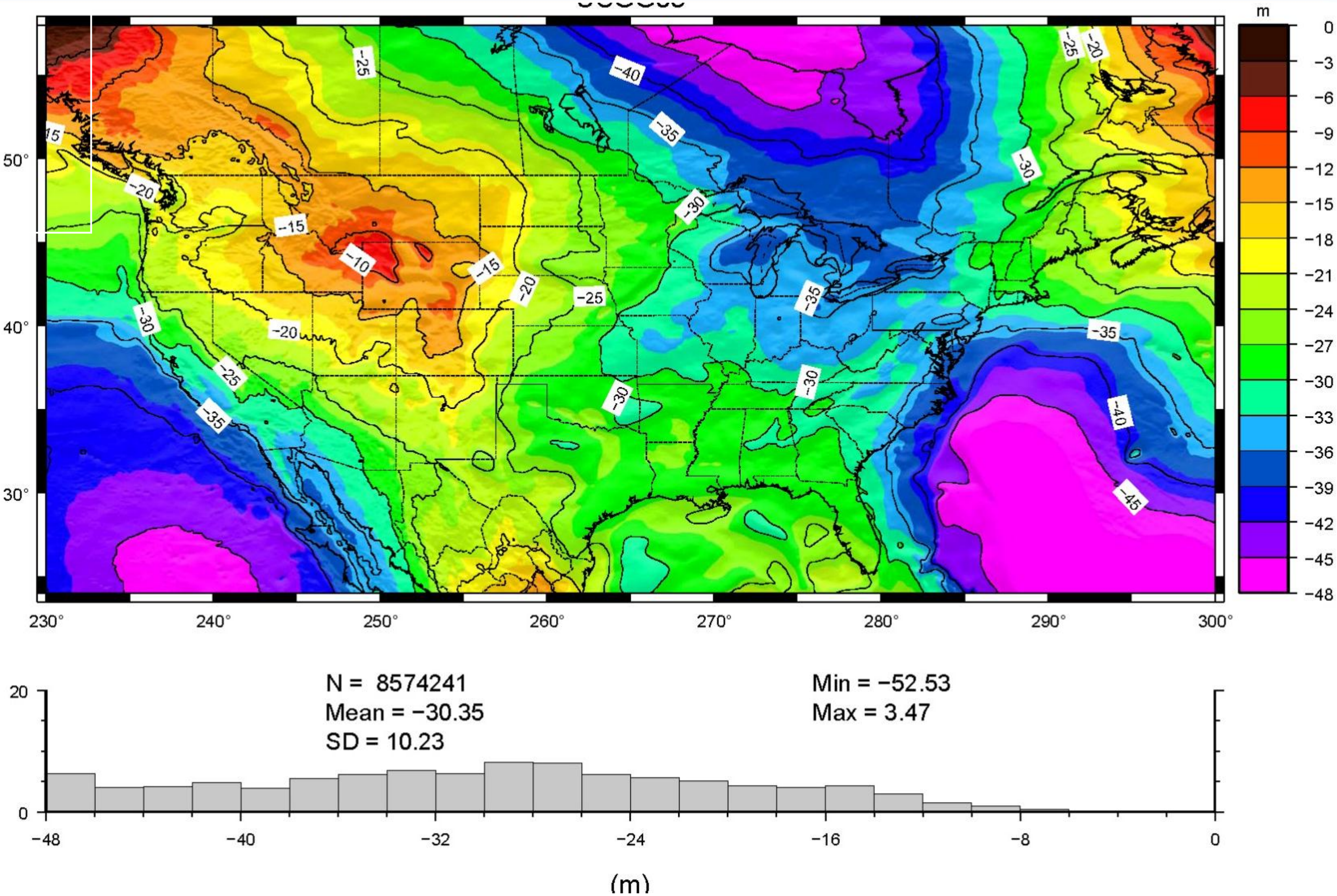
Uncertainties in EGM2008



Taken from “An Earth Gravitational Model to Degree 2160: EGM2008”, Pavlis, et al. EGU 2008.

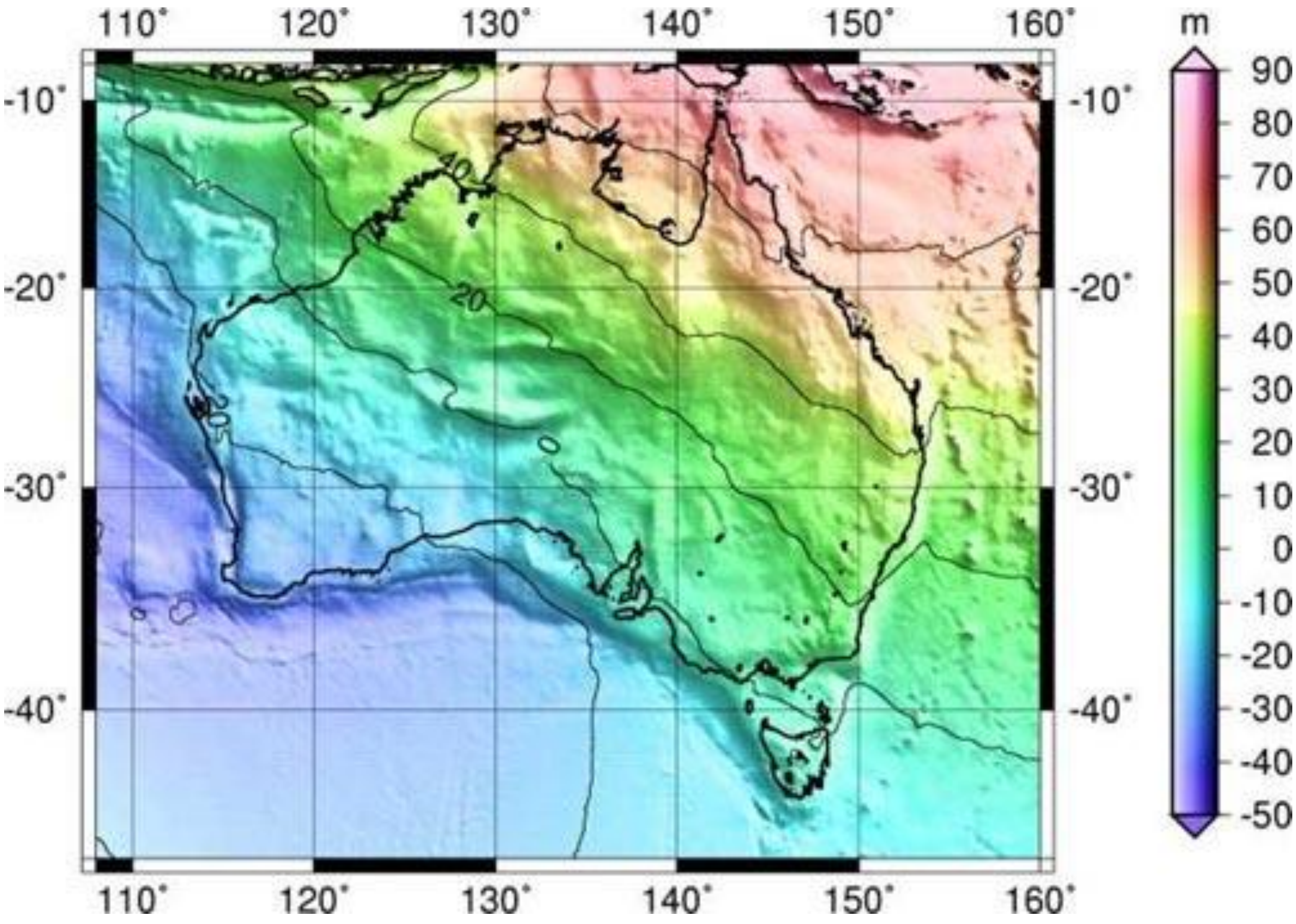


U.S. Gravimetric Geoid 2009



Ausgeoid98 Geoid in Australia

Geoid Height Contours on the GRS80 RE



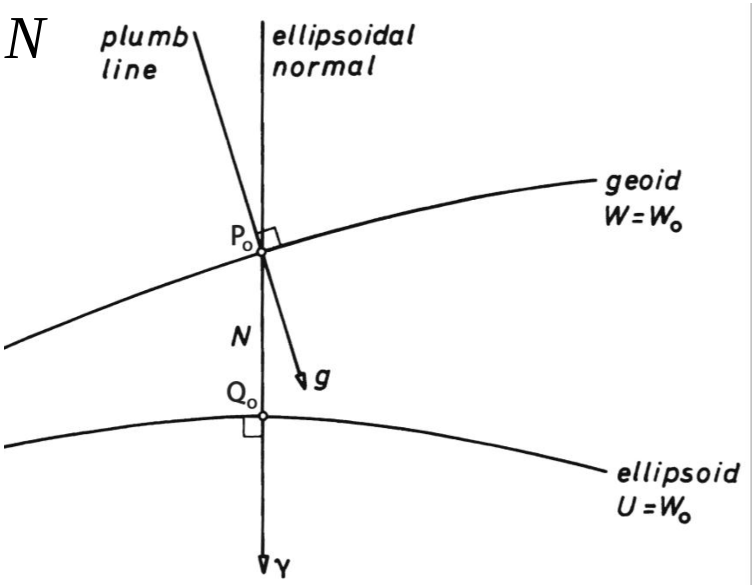
Geoid, Quasi-Geoid & Solution of the GBVP

Linking the "(Real)World" and "(Normal)Model" in Geodesy

- Approximation of the **shape**, **size** and **gravity field of the Earth** by **those of an equipotential ellipsoid of revolution** (*Reference Ellipsoid*) with the same mass and same rotational velocity as the Earth, i.e. GRS80

$$W = U + T \quad ; \quad g = \gamma + \delta g \quad ; \quad H = h - N$$

- T is the disturbing potential
- Brun's equation: $N = \frac{T}{\gamma}$
- Gravity disturbance: $\delta g = g_{P_0} - \gamma_{P_0}$
- Gravity anomaly: $\Delta g = g_{P_0} - \gamma_{Q_0}$



L. Sanchez, 11th Int. School of the Geoid Service: heights and height datum, Loja, Ecuador, 7-11 October 2013

- Fundamental equation of physical geodesy:

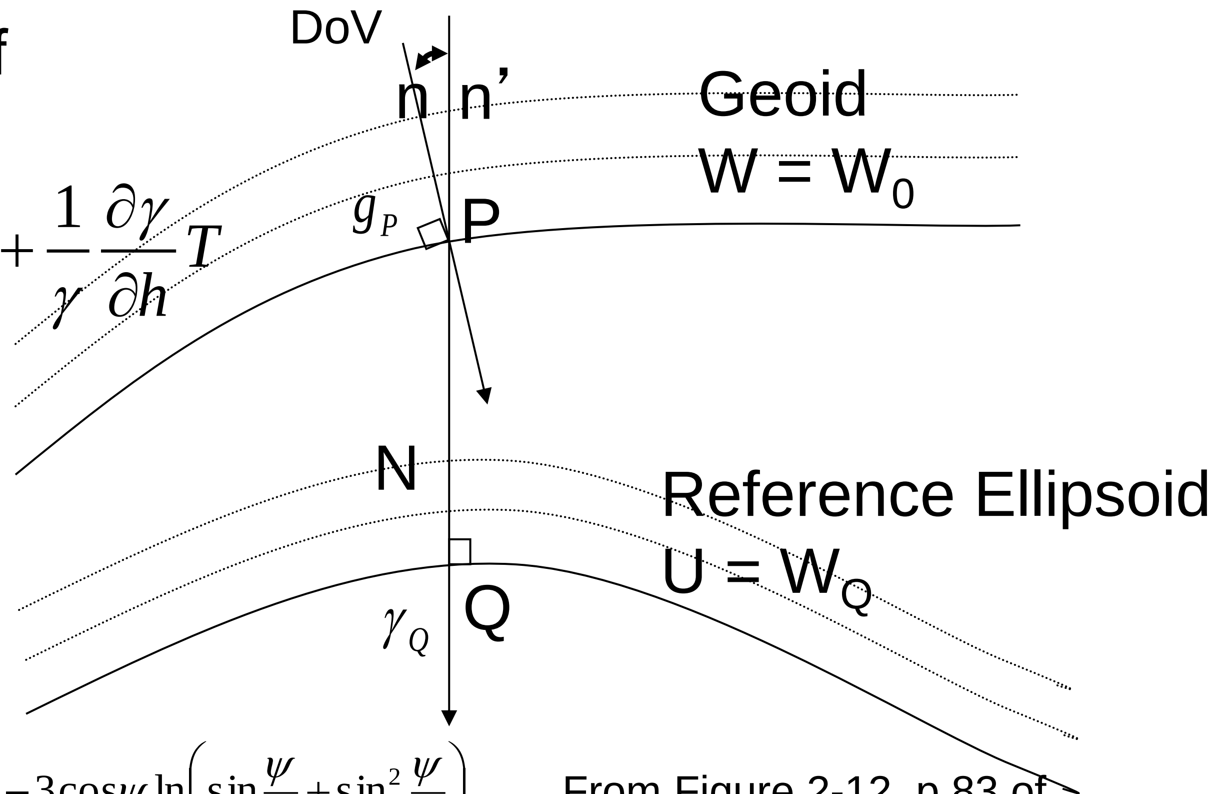
$$\Delta g = \delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}T = -\frac{\partial T}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}T$$

- Stokes' formula:

$$N = \frac{R}{4\pi G}\iint_{\sigma} \Delta g S(\psi) d\sigma$$

where:

$$S(\psi) = \frac{1}{\sin\left(\psi/2\right)} - 6\sin\left(\frac{\psi}{2}\right) + 1 - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)$$



From Figure 2-12, p.83 of Heiskanen and Moritz, 1967, *Physical Geodesy*

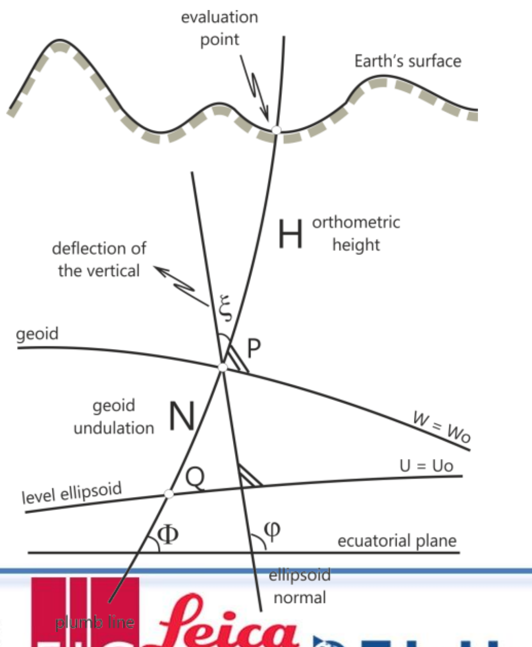
L. Sanchez, 11th Int. School of the Geoid Service: heights and height datum, Loja, Ecuador, 7-11 October 2013

Theory of Stokes (1849)

Definition: determine T, outside the geoid, from gravity anomalies *on the geoid* that satisfy the fundamental equation of physical geodesy

Boundary surface: the **geoid**

Approximation surface: the **ellipsoid**

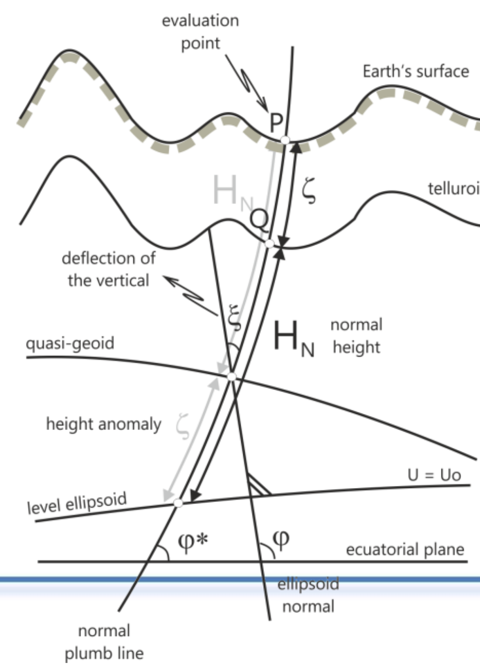


Theory of Molodenskii (1945)

Definition: determine T, outside Earth's surface, from gravity anomalies *on the Earth's surface* that satisfy fundamental equation of physical geodesy

Boundary surface: **Earth's surface**

Approximation surface: the **telluroid**



Sponsors:

Theory of Stokes (1849)

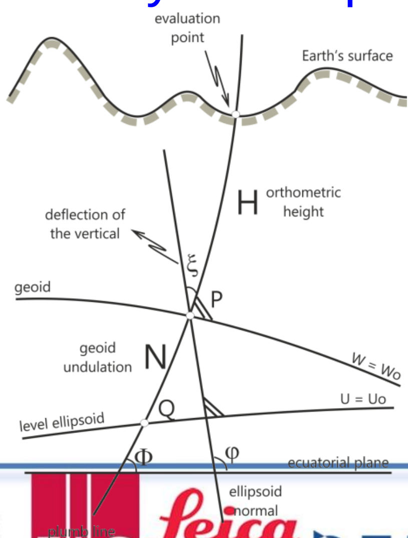
Gravity anomalies:

$$\Delta g = g_P - \gamma_Q = g_P + R_{FA} + R_B + \dots$$

⇒ P: on the geoid

⇒ Q: on the ellipsoid

Hypotheses: Geoid encloses all masses, gravity reductions for vertical gradient (Free Air) and inhomogeneous mass density are required



Theory of Molodenskii (1945)

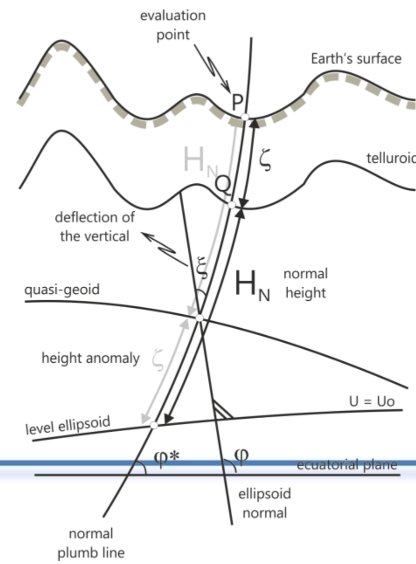
Gravity anomalies:

$$\Delta g = g_P - \gamma_Q$$

⇒ P: on the Earth's surface

⇒ Q: on the telluroid

Hypotheses: None



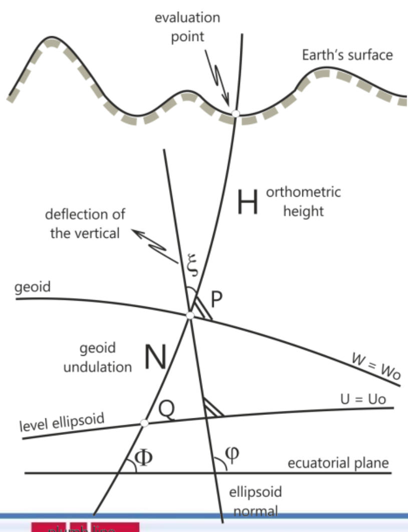
Theory of Stokes (1849)

Disturbing potential:

$$T = \frac{1}{4\pi R} \iint_{Earth} S(\psi)\Delta g \, d\sigma$$

$$S(\psi) = \frac{1}{\sin(\psi/2)} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)$$

Stokes' solution includes only 1st term of Molodenskii solution



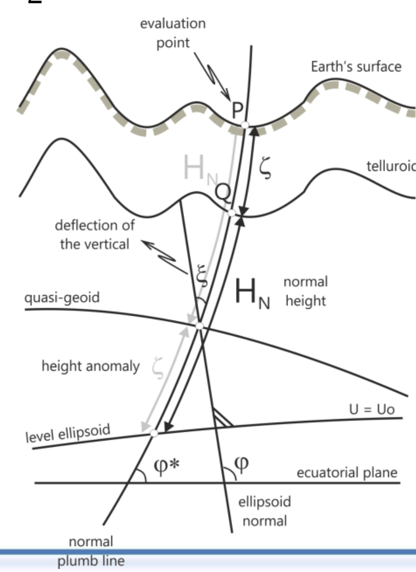
Theory of Molodenskii (1945)

Disturbing potential:

$$T = \sum_{0}^{\infty} T_k = \frac{1}{4\pi R} \oiint_{Earth} S(\psi)(\Delta g + G_1 + G_2 + \ldots) d\sigma$$

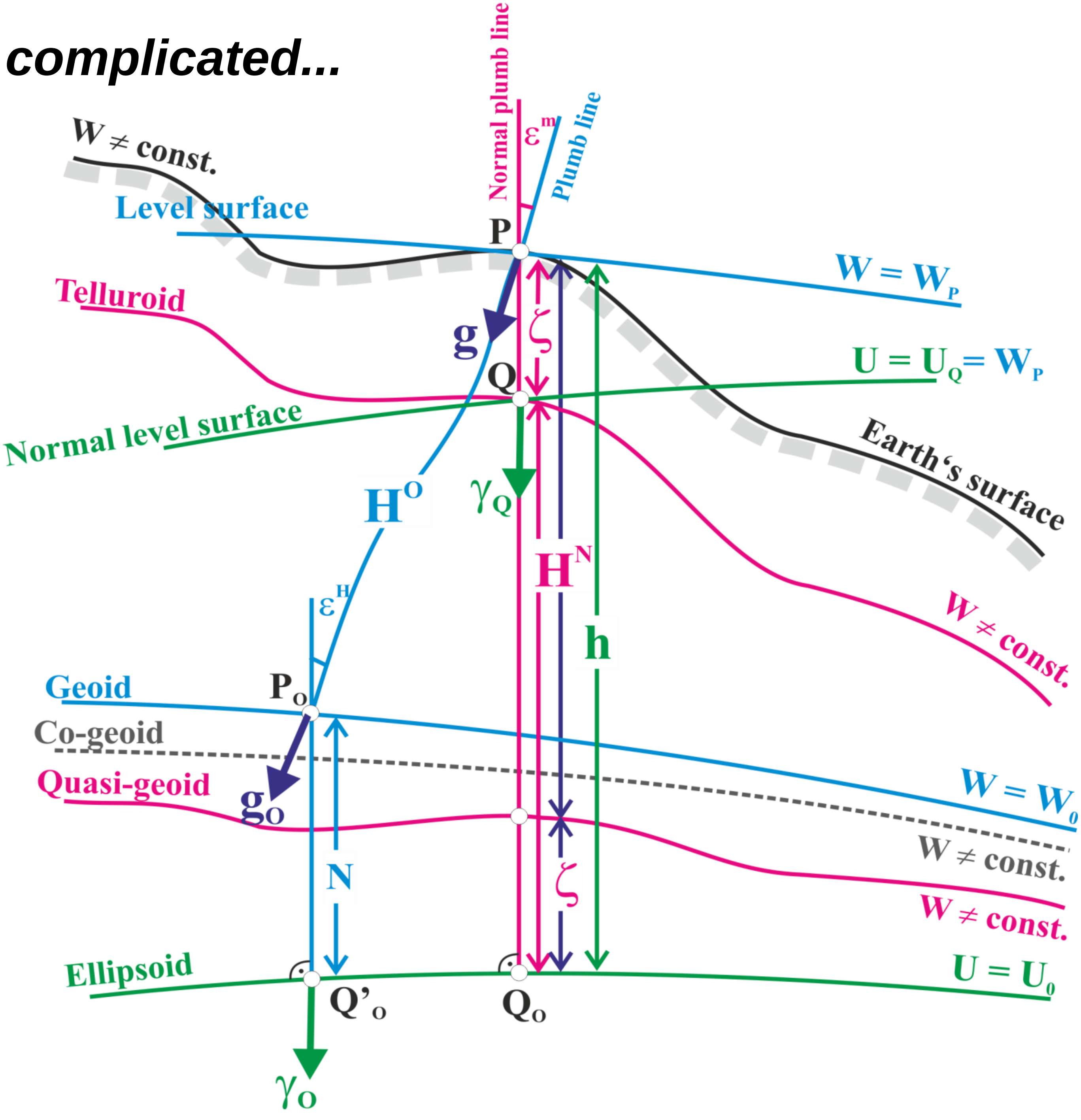
$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3}\left(\Delta g + \frac{3G}{2R}\zeta_0\right) d\sigma \approx \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \Delta g \, d\sigma$$

$$l_0 = 2R\sin\frac{\psi}{2}$$



Sponsors:

Looks complicated...



Geoid vs. Quasi-geoid

On ocean areas, geoid and quasi-geoid are the same, i.e. $H^O = H^N = 0$

On land areas, geoid and quasi-geoid differ by: $N - \zeta = \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H^O = H^N - H^O$

$\bar{g}$ **mean real gravity** value along the plumb line between Earth's surface and geoid

$\bar{\gamma}$ **mean normal gravity** value along the normal plumb line between ellipsoid and telluroid (or between Earth's surface and quasi-geoid)

$\bar{g} - \bar{\gamma}$ is a good approximation to the **Bouguer anomaly**

Geoid-Based Vertical Datums

Geoid-Based Vertical Datums

1) **Objective**: to define the vertical reference surface by a geoid model
2) **Strategy**: to compute the *best possible* geoid by solving the geodetic boundary value problem (GBVP)
3) **Advantages**:
   - No more levelling
   - Accessibility to the vertical datum using GNSS heighting
4) **Disadvantages**:
   - Low reliability in areas with poor gravity data coverage
   - Lower relative accuracy in height differences over short areas in comparison with levelling
   - Must minimise GNSS heighting errors (e.g. long occupation sessions, high precision post-processing, etc)
   - *How to access the vertical datum without GNSS equipment? Using survey equipment...*
   - *Vertical datums continue being local*

Geoid Derived From GNSS & Bench Marks

Assuming that a levelling-based height system is available, then **GNSS-levelling (quasi-)geoid heights** can be determined by making GNSS-derived ellipsoidal heights on levelled bench marks satisfy condition:

$$N_{GNSS/levelling} = h_{GNSS} - H_{levelling}$$

This information can be used to **correct** the national geoid model (whether gravimetric or GGM) to account for any bias between geoid surface and zero height surface, ensuring closure of h-N-H=0 relation... *see later slides, e.g. Ausgeoid09*

Some Comments

- Such "geoid correction models" or "hybrid geoids" preserve **heritage** vertical datums... *ensuring new GNSS-derived heights are consistent with old maps and geospatial data:*

$$\hat{H}_{GNSS} = \hat{h}_{GNSS} - N_{correction\ model}$$

- Alternatively a new **geoid-based vertical datum** may be defined to use with GNSS, *creating orthometric-like heights from observed ellipsoidal heights and a geoid model:*

$$\hat{H}_{GNSS} = \hat{h}_{GNSS} - N_{geoid\ model}$$

- Care must still be taken to ensure geoid potential is W_o, and compatible with gravity field

GNSS/Geoid-Based Height Systems: Examples

- Canada has introduced the new geoid-based height system **CGVD2013**
- USA plans to introduce a new geoid-based height system in 2022... *To improve the quality of the gravimetric geoid model they are remeasuring gravity over the whole country using airborne techniques in the* ***GRAV D*** *project*
- New Zealand is also modernising its vertical datum using this approach

Name:	Canadian Geodetic Vertical Datum of 2013
Abbreviation:	CGVD2013
Type of datum:	Gravimetric (geoid)
Vertical datum:	W_0 = 62,636,856.0 m^2s^{-2}
Realisation:	Geoid model CGG2013 (ITRF2008 and NAD83(CSRS))
Type of height :	Orthometric

Canadian Gravimetric Geoid of 2013 (CGG2013)

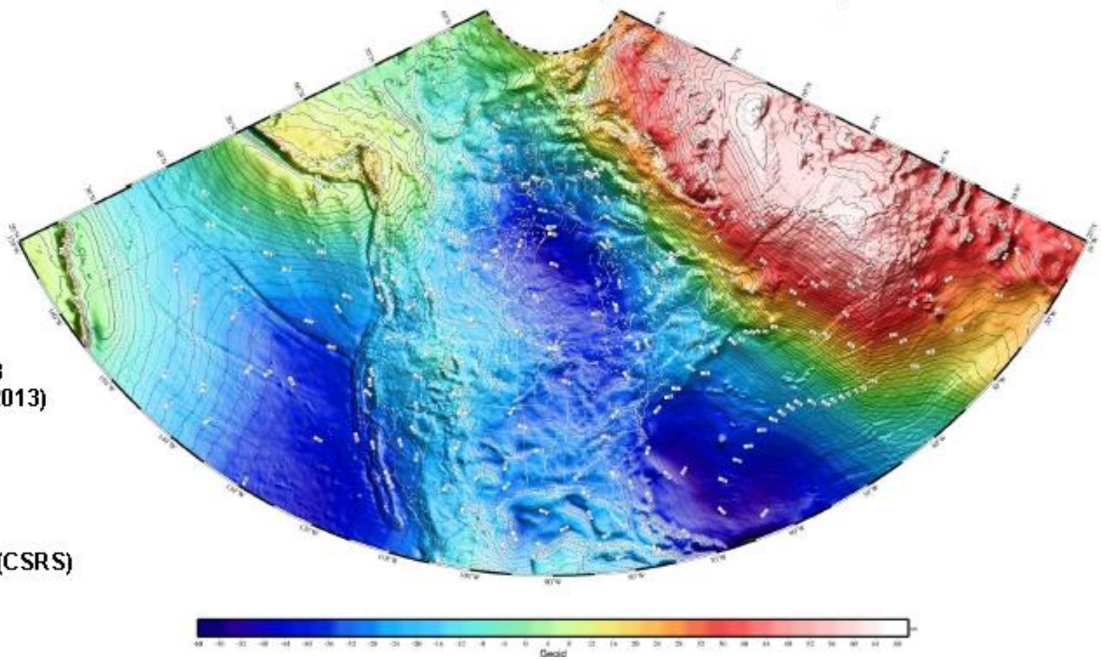
Boundaries
North: 90°
South: 10°
West: -170°
East: -10°

Resolution
2' x 2'

Satellite model
EIGEN-6C3stat (GFZ)
Förste et al., IAG 2013
GOCE (until May 24, 2013)

Transition zone
Degrees: 120-180

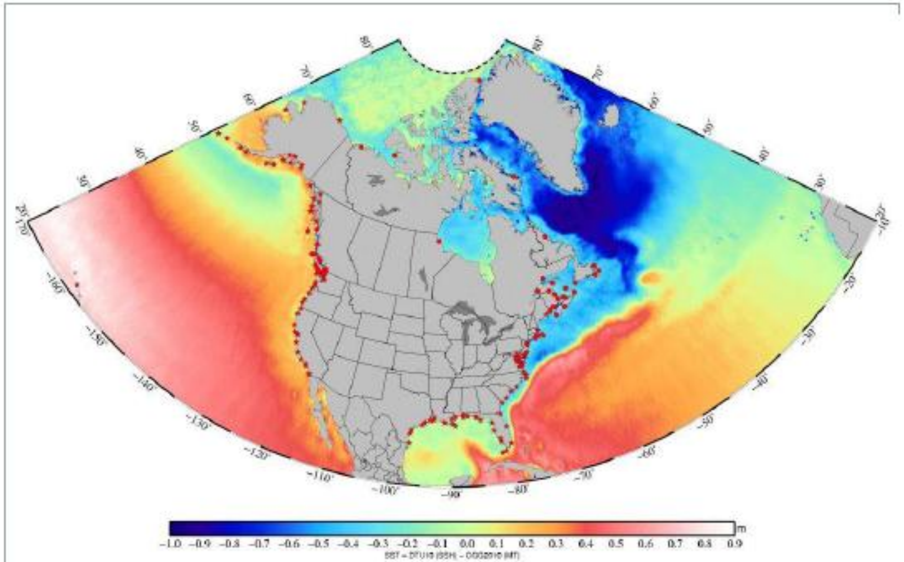
Reference frames
ITRF2008 and NAD83(CSRS)



Véronneau and Huang (2014)

WHAT is the definition of CGVD2013?

- CGVD2013: Conventional equipotential surface (W_0 = 62,636,856.0 m^2/s^2) averaging the coastal mean sea level for North America measured at Canadian and American tide gauges.



- U.S. NGS and NRCan's GSD signed an agreement (16 April 2012) to realize and maintain a common vertical datum for USA and Canada defined by W_0 = 62,636,856.0 m^2/s^2

- It also corresponds to the current convention adopted by the International Earth Rotation and Reference Systems Service (IERS) and International Astronomical Union (IAU).

- Canada's recommended definition for a World Height System

Natural Resources Canada / Ressources naturelles Canada

Canada

Véronneau and Huang (2014)

Geoid-Based Vertical Datums: Open Questions

- Geoid computation is not a unified/standardised procedure
- In a similar way, GNSS positioning under different conditions or processed with different strategies produces different heights
- If new gravity data and new analysis strategies are available, it is usual to compute improved geoid models. *How frequently shall the vertical datum be updated? Is it convenient for the users to change height values regularly*?
- The state-of-the-art allows the computation of geoid models with a maximum accuracy in dm-level. This could satisfy some practical applications, *but what about measuring, understanding and modelling global change effects?*
- The relationship $h = H + N$ is applied today to estimate the reliability of the different heights derived from independent methods. *If H from levelling does not exist anymore, how can we verify the reliability of $H=h-N$?*