Transformation of Conformal Coordinates of Type Lambert Conic from a Global Datum, (ITRS) to a Local Datum (regional, National)

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Key words: Global geodetic datum, ITRS, Local geodetic datum, Lambert conic projection, Curvilinear datum transformation.

SUMMARY

Local geodetic systems with appropriate choices of map projections have been developed in the past, in order to satisfy the surveying and mapping requirements of countries all over the earth. In most of these geodetic systems, the horizontal and vertical points are separated in location, monumentation, and measurement, and are referred to different datums. Local horizontal geodetic points have approximate or no heights, and benchmarks have very weak horizontal coordinates. In recent years, a growing trend toward the use of GPS observations and global mapping satellite systems have resulted in position based products in a world reference frame. There is therefore need in geodetic practice to transform coordinates referred to the local geodetic system to the global geodetic system and vice versa. Also a fundamental activity in land surveying is the integration of multiple sets of geodetic data, gathered in various ways, into a single consistent data set, that is into a common geodetic reference frame. Thus geodetic coordinate transformations find application in several instances, such as navigation, revision of older maps, cadastral surveying, GIS, industrial surveying, deformation studies, geo-exploration, etc.

This paper treats the problem of how to transform from global datum, for instance from the International Terrestrial Reference system (ITRS) to a local datum, for instance of type regional or national, for the practical case of the Lambert projection of the sphere or the ellipsoid-of-revolution to the cone. We design the two projection constants $n(\varphi_1, \varphi_2)$ and $m(\varphi_1)$ for the Universal Lambert Conic project of the ellipsoid-of-revolution. The task to transform from a global datum with respect to the ellipsoid-of-revolution $E_{a,b}^2$ to local datum with respect to the alternative ellipsoid-of-revolution $E_{a,b}^2$, without local ellipsoidal height, is solved by an extended numerical example.
1. INTRODUCTION

The Lambert projection of the sphere to the cone is one of the most popular map projections. Here we generalize it to the ellipsoid-of-revolution and study in detail the equations which govern a datum transformation extend by form parameters. The inclusion of form parameters is needed when we switch from a global datum, for instance from the International Terrestrial Reference System (ITRS) to a local datum (regional, National) and vice versa.

Section 2 of this paper is devoted to generalize the classical Lambert conic project of the sphere to the ellipsoid-of-revolution and introduce the datum transformation given in the appendix as a curvilinear transformation characterized by seven parameters (three for translation, three for rotation, one or scale).

At first, we introduce by Definition 1.1 the Universal Lambert Conic Projection of the ellipsoid-of-revolution in local coordinates \((\tilde{x}, \tilde{y})\) or \((\tilde{a}, r)\). The latitudes of two parallel circles are mapped equidistantly. They are referred as two projection constants of the first kind and of the second kind respectively, here called \(n(\varphi_1, \varphi_2)\) and \(m(\varphi_1)\).

Secondly, in order to transform Lambert conformal coordinates given in a global datum with respect to the ellipsoid-of-revolution \(E^2_{A,B}\) into Lambert conformal coordinates in a local datum with respect to the ellipsoid-of-revolution \(E^2_{a,b}\) we take advantage of the Taylor expansion of second order. Given ellipsoidal coordinates of a point by means of \(\{\Lambda, \Phi, \eta\}\), for instance by GPS, GLONASS or others, we design the corresponding matrices \(K\) and \(A\) which build up the 7 datum parameters and the 2 form parameters of transformation to a local datum. We collect the various results in section 2.1 – section 2.3.

The third section is based on a detailed example. We start from seven datum parameters, ellipsoidal parameters, and from two values of latitudes of types parallel circles in Tables 1a and 1b. Global and local sets of coordinates \(\{\Lambda, \Phi, \eta\}\), and \((\lambda, \varphi)\) are represented by Table 2 and Table 3, while datasets of type Eastings and Northings of 14 points are given in Table 4. Table 5 contains the computed values of Eastings and Northings based on equation (11) and (12) without including the second order terms and their corresponding difference \((dE, dN)\). Their differences are in the centimeter range and therefore cannot be neglected.

We took care of our global representation of ellipsoidal heights in terms of orthonormal functions (Grafarend and Engels 1992). The datum transformations including the form parameters are referred to contributions of Grafarend, Krumm and Okeke (1995), Grafarend.

2. EQUATION GOVERNING DATUM TRANSFORMATION EXTENDED BY FORM PARAMETERS OF THE LAMBERT CONIC PROJECTION (CONFORMAL)

Definition 1.1 (Universal Lambert Conic Projection, local coordinates of $E_{a,b}^2$ on $\Box_{a,b}^2$): A conformal transformation of ellipsoidal coordinates of type “surface normal” ellipsoidal longitude $l$ and “surface normal” ellipsoidal latitude $\varphi$ into Cartesian coordinates $\tilde{x}$, $\tilde{y}$ with respect to a local ellipsoid of revolution $E_{a,b}^2$ is called Universal Lambert Conic Projection if

\begin{align*}
\tilde{x} &= r \cos \tilde{\alpha}, \quad \tilde{y} = r \sin \tilde{\alpha} \\
\partial \tilde{\varphi} &= nl \\
r &= am[\tan(\pi/4 - \varphi/2)(1 + e \sin \varphi)^{\sqrt{2}}] =: r(\varphi)
\end{align*}

subject to

\begin{align*}
m &= m(\varphi) := \cos \varphi \left[ n/\sqrt{1 - e^2 \sin^2 \varphi} \right] \left[ \tan(\pi/4 - \varphi/2)(1 + e \sin \varphi)^{\sqrt{2}} \right]^{-n} \\
n &= n(\varphi_1, \varphi_2) := \left[ \ln[\cos \varphi_1(1 - e^2 \sin^2 \varphi_1)^{1/2}] - \ln[\cos \varphi_1(1 - e^2 \sin^2 \varphi_2)^{1/2}] \right] \\
&\quad \left[ \ln[\tan(\pi/4 - \varphi_1/2)(1 + e \sin \varphi_1)^{\sqrt{2}}] - \ln[\tan(\pi/4 - \varphi_2/2)(1 + e \sin \varphi_2)^{\sqrt{2}}] \right]^{-1/2}
\end{align*}

holds. We have denoted by $a$ the semi-major axis, by $b$ the semi-minor axis, by $e \sqrt{1 - b^2/a^2}$ the relative eccentricity of $E_{a,b}^2$. $(\lambda, \varphi)$ are the curvilinear coordinates which cover the elliptic cone $\Box_{a,b}^2$ which is developed onto $\mathbb{R}^2$ covered by its polar coordinates $(\vartheta, \varphi)$. $(\varphi_1, \varphi_2)$ are the latitudes of those parallel circles (coordinate lines $\varphi_1 =$const and $\varphi_2 =$const) which are mapped equidistantly. $n(\varphi_1, \varphi_2)$ and $m(\varphi_1)$ are conveniently called projection constants of the first kind and of the second kind, respectively.

In order to transform Lambert conformal coordinates given in a global datum with respect to the ellipsoid of revolution $E_{a,b}^2$ into Lambert conformal coordinates in a local datum with respect to the ellipsoid of revolution $E_{a,b}^2$ we take advantage of the Taylor expansion of second order.
\[ \lambda = \Lambda + \delta \lambda, \ \varphi = \Phi + \delta \Phi, \ a = A + \delta A, \ e = E + \delta E \]

so that

\[ \tilde{\alpha}(\lambda, e) = \tilde{\alpha}(\Lambda + \delta \lambda, E + \delta E) = \tilde{\alpha}(\Lambda, E)\Lambda + \frac{d\tilde{\alpha}}{d\lambda} \delta \lambda + \frac{d\tilde{\alpha}}{de} \delta E + O_{2\epsilon} \]

\[ r(\varphi, a, e) = r(\Phi + \delta \Phi; A + \delta A, E + \delta E) = r(\Phi; A, E) + r_\phi \delta \Phi + r_a \delta A + r_e \delta E + O_{2\epsilon} \]

or

\[ \tilde{x} = r \cos \tilde{\alpha} = r(\Phi + \delta \Phi; A + \delta A, E + \delta E) \cos \tilde{\alpha}(A + \delta A, E + \delta E) \]

\[ \tilde{y} = r \sin \tilde{\alpha} = r(\Phi + \delta \Phi; A + \delta A, E + \delta E) \sin \tilde{\alpha}(A + \delta A, E + \delta E) \]

\[ \tilde{x} = r(\Phi; A, E) \cos \tilde{\alpha} - r(\Phi; A, E)n \sin \tilde{\alpha} \delta \lambda + \]

\[ + r_\phi(\Phi; A, E) \cos \tilde{\alpha} \delta \Phi + r_a(\Phi; E) \cos \tilde{\alpha} \delta A + \]

\[ + (-r(\Phi; A, E) \sin \tilde{\alpha} \Lambda_n + \cos \tilde{\alpha} \Lambda_r) A \delta \lambda + O_{2\epsilon} \]

\[ O_{2\epsilon}(\Phi; A, E) = \left( -\frac{r \sin \tilde{\alpha} \Lambda_n}{A} + \frac{\cos \tilde{\alpha} \Lambda_r}{A} \right) \delta (A, E) + \frac{\cos \tilde{\alpha} r_\phi}{A} \delta (A, \Phi) + \frac{-rn \sin \tilde{\alpha}}{A} \delta (A, \Lambda) \]

\[ -(\cos \tilde{\alpha} n r \Lambda_n + r \sin \tilde{\alpha} n_r + n \sin \tilde{\alpha} r_\phi) \delta (A, E) + n \sin \tilde{\alpha} r_\phi \delta (\Lambda, \Phi) - \frac{1}{2} n^2 r \cos \tilde{\alpha} \delta \lambda^2. \]

\[ \tilde{y} = r(\Phi; A, E) \sin \tilde{\alpha} + r(\Phi; A, E)n \cos \tilde{\alpha} \delta \lambda + r_\phi(\Phi; A, E) \sin \tilde{\alpha} \delta \Phi + \]

\[ r_a(\Phi; E) \sin \tilde{\alpha} \delta A + (r(\Phi; A, E) \cos \tilde{\alpha} \tilde{\alpha}_r + \cos \tilde{\alpha} r_\phi(\Phi; A, E)) \delta \Phi + O_{2\epsilon} \]

\[ O_{2\epsilon}(\Phi; A, E) = \left( -\frac{r \cos \tilde{\alpha} \Lambda_n}{A} + \frac{\sin \tilde{\alpha} \Lambda_r}{A} \right) \delta (A, E) + \frac{\sin \tilde{\alpha} r_\phi}{A} \delta (A, \Phi) + \frac{-rn \cos \tilde{\alpha}}{A} \delta (A, \Lambda) \]

\[ -(\sin \tilde{\alpha} n r \Lambda_n + r \cos \tilde{\alpha} n_r + n \cos \tilde{\alpha} r_\phi) \delta (A, E) + n \cos \tilde{\alpha} r_\phi \delta (\Lambda, \Phi) - \frac{1}{2} n^2 r \sin \tilde{\alpha} \delta \lambda^2. \]

Where

\[ \tilde{\alpha} = n(E) \Lambda; \ \tilde{\alpha}_r(\Lambda, E) := \frac{d\tilde{\alpha}}{de}(\Lambda, E) = \Lambda n_r(\Lambda, E), \ \tilde{\alpha}_r(E) := \frac{dn}{de}(E) \]

These expressions up to second order \( O_{2\epsilon} \) or \( O_{2r}, O_{2s} \) include the first derivatives

\[ r_\phi(\Phi; A, E) := \frac{dr}{d\varphi}(\Phi; A, E) \]

\[ r_a(\Phi; E) := \frac{dr}{dA}(\Phi; E) \]

\[ r_\epsilon(\Phi; A, E) := \frac{dr}{de}(\Phi; A, E) \]
as well as the longitude increments \( \delta \Lambda \), latitude increments \( \delta \Phi \), increment of semi-major axis \( \delta A \) and of eccentricity \( \delta E \). The zero order terms includes the radial functions \( r(\Phi; A, E) \) with respect to the global semi-major axis \( A \) and global eccentricity \( E \). The projection constants \( m(\phi), n(\phi_1, \phi_2) \) are fixed under a datum transformation by means of \( \phi_1 = \Phi_1, \phi_2 = \Phi_2 \). In addition, second order terms containing second order variation of the constant terms \( m(\phi; E), n(\phi_1, \phi_2, E) \) with respect to the eccentricity \( E \) are neglected.

**Theorem 1.2** (Datum transformation, Universal Lambert Conic Projections of \( E_{a,b}^2 \) towards \( E_{A,B}^2 \)): Let the seven datum parameters of type translations, rotations, scale and two form parameters of type semi-major axis, relative eccentricity between a local and global reference system be given. Then polar ULC coordinates transform according to (16), (17) while Cartesian ULC coordinates transform according to (21), (22).

We shall be able to switch from the global ULC coordinates (polar or Cartesian), to the local ULC coordinates polar or Cartesian) or vice versa.

### 2.1 First Variation Of The Radial Coordinate \( r(\phi; a, e) \)

The first variation of the radial coordinate is given by equation (13i)

\[
(13i) \quad r_\phi = amn(r1r2)
\]

Where:

\[
r1 := \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right); \quad r2 := \left(1 + e \sin \phi\right)^{e/2}, \quad \frac{d(r1r2)}{d_\phi} = \frac{-r2}{2} - \frac{r1^2 r2}{2} + \frac{e^2 r1r2 \cos \phi}{1 - e^2 \sin^2 \phi}
\]

\[
(14i) \quad r_a = m\left[\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\left(1 + e \sin \phi\right)^{e/2}\right]^\phi
\]

\[
(15i) \quad re = r\left(\frac{\phi}{c} + \ln(r1r2)n_e + \frac{n_2}{r2}\right)
\]

Subject to

\[
\frac{dr^2}{de} = \frac{\ln(r2^{2/c})r2}{2} + \frac{er2 \sin \phi}{1 - e^2 \sin^2 \phi}
\]

\[
c = am; \quad c2 := (1 - e^2 \sin^2 j)^{1/2}; \quad c3 := \tan\left(\frac{\pi}{4} - \frac{j}{2}\right); \quad c4 := \left(\frac{1 + e \sin j}{1 - e \sin j}\right)^{e/2}
\]

\[
\frac{dc}{de} = c\left(\frac{\phi}{c2} - \frac{d_{c1}}{c2} - \ln(c3c4)n_e - \frac{n_e}{n}\right), \quad \frac{dc2}{de} = \frac{e \sin^2 \phi_1}{n}, \quad \frac{dc3}{de} = \frac{\ln(c4^{2/c4})}{2} + \frac{ec4 \sin \phi}{1 - e^2 \sin^2 \phi_1}
\]
Subject to
\[ nu = \ln[\cos \varphi_2 (1 - e^2 \sin^2 \varphi_1)^{1/2}] - \ln[\cos \varphi_1 (1 - e^2 \sin^2 \varphi_2)^{1/2}] \]
\[ nl = \ln[\tan(\frac{\pi}{4} - \frac{\varphi_1}{2})(1 + e \sin \varphi_1)^{\nu/2}] - \ln[\tan(\frac{\pi}{4} - \frac{\varphi_2}{2})(1 + e \sin \varphi_2)^{\nu/2}] \]
\[ \frac{dn}{de} = \frac{e \sin^2 \varphi_1}{e^2 \sin \varphi_1 - 1} - \frac{e \sin^2 \varphi_2}{e^2 \sin \varphi_2 - 1} \]

With the first derivatives of the radial polar coordinate \(r(\varphi; a, e)\) given in section 2.1 we are able to initiate the 7-parameter datum transformation reviewed in the Appendix, namely by implementing the \textit{curvilinear datum transformation} \(\delta \Lambda (A17i), \delta \Phi (A17ii)\), a procedure summarized in sections 2.2 and 2.3.

### 2.2 Datum Transformation of Universal Lambert Conic Projection (ULC) - Polar form

Let \( \tilde{\alpha} \) be given as:
\[ \tilde{\alpha} = n(E) \Lambda \frac{d\tilde{\alpha}}{d\Lambda} + \frac{d\tilde{\alpha}}{d\varphi} \varphi. \]

Then
\[ (16) \alphatilde = n(E)\Lambda + n(E)n(E)(k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14}\alpha + a_{15}\beta + a_{16}\gamma + a_{17}s) + \Lambda n(E)\delta E. \]

Also
\[ (17) \quad r = R(\Phi; A, E) + r_0 \delta \Phi + r_0 \delta A + r_0 \delta \varphi E + O_{2r}, \]
\[ (17i) \quad \delta \Lambda = k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14}\alpha + a_{15}\beta + a_{16}\gamma + a_{17}s \]
\[ (17ii) \quad \delta \Phi = k_2 + a_{21}t_x + a_{22}t_y + a_{23}t_z + a_{24}\alpha + a_{25}\beta + a_{26}\gamma + a_{27}s \]
The array elements of \(K\) and \(A\) are collected in the appendix.

\[ (18) \quad r = R + r_0 (k_2 + a_{21}t_x + a_{22}t_y + a_{23}t_z + a_{24}\alpha + a_{25}\beta + a_{26}\gamma + a_{27}s) + r_0 \delta A + r_0 \delta \varphi E \]

### 2.3 Datum Transformation of Universal Lambert Conic Project (ULC)-Cartesian form

The equations for datum transformation of ULC, Cartesian form, are given in equations (19) and (20) below. Note again that the array elements of \(K\) and \(A\) are collected in the appendix.

\[ (19) \quad \tilde{x} = \tilde{X} - r(\Phi)n \sin \tilde{\alpha} \delta \Lambda + r_0 \cos \tilde{\alpha} \delta \Phi + r_0 \cos \tilde{\alpha} \delta A + (r \sin \tilde{\alpha} \tilde{\varphi}, r_0 \cos \tilde{\alpha} \delta \varphi) \varphiE \]
\[ (20) \quad \tilde{y} = \tilde{Y} + r(\Phi)n \cos \tilde{\alpha} \delta \Lambda + r_0 \sin \tilde{\alpha} \delta \Phi + r_0 \sin \tilde{\alpha} \delta A + (r \sin \tilde{\alpha} \tilde{\varphi}, r_0 \cos \tilde{\alpha} \delta \varphi) \varphiE \]
\[ x = \tilde{X} - n\tilde{Y} (k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14} \alpha + a_{15} \beta + a_{16} \gamma + a_{17} s) \]  
(21) \[ r_y (\Phi; A, E) \cos(n \Lambda) k_2 + a_{21} t_x + a_{22} t_y + a_{23} t_z + a_{24} \alpha + a_{25} \beta + a_{26} \gamma + a_{27} s + \]
\[ r_y (\Phi; E) \cos(n \Lambda) \delta A + (r \sin(n \Lambda) \alpha \sin(n \Delta) \sin(n \Delta) \delta \Phi) \]
\[ y = \tilde{Y} - n\tilde{X} (k_1 + a_{11}t_x + a_{12}t_y + a_{13}t_z + a_{14} \alpha + a_{15} \beta + a_{16} \gamma + a_{17} s) \]  
(22) \[ r_y (\Phi; A, E) \sin(n \Lambda) k_2 + a_{21} t_x + a_{22} t_y + a_{23} t_z + a_{24} \alpha + a_{25} \beta + a_{26} \gamma + a_{27} s + \]
\[ r_y (\Phi; E) \sin(n \Lambda) \delta A + (r \cos(n \Lambda) \alpha \sin(n \Delta) \sin(n \Delta) \delta \Phi) \]

3. RESULTS OF NUMERICAL EXAMPLE

Assume we have measured the ellipsoidal coordinates of a point by means of \{\Lambda, \Phi, \Upsilon\} for instance by satellite positioning technology of type GPS, GLONAS or others, it is to be noted that for the synthesis of the “design vector \( \mathbf{K} \)” as well as for the design matrix \( \mathbf{A} \)” we need the global ellipsoidal height. Secondly we have to get information of the variation of the 7 datum parameters and the 2 form parameters, namely about the basic data which establish a local and a global UMP chart.

Table 1a lists the datum transformation parameters between a set of global coordinates Table 2) and the corresponding local coordinates (Table 3), while Table 1b lists their corresponding ellipsoidal parameters. Eastings and Northings computed from equations (1) and (2) are given in Table 4. To illustrate the numerical precision of the transformation equations developed in this paper, Eastings and Northings are again computed via equations (11) and (12), once without the second order terms (Table 5), and again including the second order terms (Table 6). The computed Root Mean Square (RMS) of the discrepancies in Table 5 for the Eastings and Northings are 0.0623m and 0.0127m respectively, indicating differences with the original data in centimeter range. On the other hand, the RMS discrepancies in Table 6 are 0.0049m and 0.0029m for Eastings and Northings respectively, showing differences in millimeter.

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<thead>
<tr>
<th>Table 1a: Datum transformation parameters (DTP)</th>
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<tbody>
<tr>
<td>DTP</td>
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<tr>
<td>--------------------------------------------</td>
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<tr>
<td>Values</td>
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<table>
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<th>Table 1b: Ellipsoidal parameters (EP)</th>
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Table 2: Global coordinates

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<tr>
<th>Point No</th>
<th>Λ (deg. min. sec.)</th>
<th>Φ (deg. min. sec.)</th>
<th>H (m)</th>
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<tbody>
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<td>50 00 3.000</td>
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<tr>
<td>2</td>
<td>4 00 3.000</td>
<td>50 03 2.000</td>
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<td>3</td>
<td>5 00 1.000</td>
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<td>6</td>
<td>4 57 1.000</td>
<td>50 24 0.000</td>
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Table 3: Local coordinates

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<th>Φ (deg. Min. sec.)</th>
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Table 4: Eastings and Northings computed from equations (1) and (2)

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<th>Northing (m)</th>
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<td>8</td>
<td>5240967.14964</td>
<td>413834.06750</td>
</tr>
<tr>
<td>9</td>
<td>5241104.63748</td>
<td>314071.68189</td>
</tr>
<tr>
<td>10</td>
<td>5233031.95125</td>
<td>314071.68189</td>
</tr>
<tr>
<td>11</td>
<td>5220287.09258</td>
<td>309779.45390</td>
</tr>
</tbody>
</table>
Table 5: Eastings and Northings computed from equations (11) and (12) without the second order terms, and their differences (δ E, δ N) with that of table 4.

<table>
<thead>
<tr>
<th>Point No</th>
<th>Easting(m)</th>
<th>Northing(m)</th>
<th>δ E(m)</th>
<th>δ N(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5316191.73310</td>
<td>286842.26461</td>
<td>-0.0624</td>
<td>-0.0127</td>
</tr>
<tr>
<td>2</td>
<td>5310671.50437</td>
<td>286504.60277</td>
<td>-0.0621</td>
<td>-0.0120</td>
</tr>
<tr>
<td>3</td>
<td>5289719.52240</td>
<td>356830.63355</td>
<td>-0.0617</td>
<td>-0.0127</td>
</tr>
<tr>
<td>4</td>
<td>5264828.61644</td>
<td>277000.84813</td>
<td>-0.0621</td>
<td>-0.0124</td>
</tr>
<tr>
<td>5</td>
<td>5273283.19744</td>
<td>320050.71153</td>
<td>-0.0608</td>
<td>-0.0117</td>
</tr>
<tr>
<td>6</td>
<td>5267793.94685</td>
<td>351788.78227</td>
<td>-0.0621</td>
<td>-0.0131</td>
</tr>
<tr>
<td>7</td>
<td>5242904.34270</td>
<td>385581.97431</td>
<td>-0.0617</td>
<td>-0.0126</td>
</tr>
<tr>
<td>8</td>
<td>5240967.08702</td>
<td>413834.05540</td>
<td>-0.0626</td>
<td>-0.0120</td>
</tr>
<tr>
<td>9</td>
<td>5241104.57533</td>
<td>314071.95256</td>
<td>-0.0615</td>
<td>-0.0117</td>
</tr>
<tr>
<td>10</td>
<td>5233031.88781</td>
<td>314071.66870</td>
<td>-0.0634</td>
<td>-0.0131</td>
</tr>
<tr>
<td>11</td>
<td>5220287.02961</td>
<td>309779.44159</td>
<td>-0.0629</td>
<td>-0.0123</td>
</tr>
<tr>
<td>12</td>
<td>5175772.21001</td>
<td>307157.53512</td>
<td>-0.0639</td>
<td>-0.0124</td>
</tr>
<tr>
<td>13</td>
<td>5196843.32121</td>
<td>262762.70049</td>
<td>-0.0634</td>
<td>-0.0137</td>
</tr>
<tr>
<td>14</td>
<td>5232054.40769</td>
<td>264582.06021</td>
<td>-0.0607</td>
<td>-0.0133</td>
</tr>
</tbody>
</table>

Table 6: Eastings and Northings computed from equations (11) and (12) including the second order terms, and their differences (δ E, δ N) with that of Table 4.

<table>
<thead>
<tr>
<th>Point No</th>
<th>Easting(m)</th>
<th>Northing(m)</th>
<th>δ E(m)</th>
<th>δ N(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5316191.79052</td>
<td>286842.27479</td>
<td>-0.0052</td>
<td>-0.0026</td>
</tr>
<tr>
<td>2</td>
<td>5310671.56180</td>
<td>286504.61296</td>
<td>-0.0047</td>
<td>-0.0025</td>
</tr>
<tr>
<td>3</td>
<td>5289719.57989</td>
<td>356830.61296</td>
<td>-0.0042</td>
<td>-0.0032</td>
</tr>
<tr>
<td>4</td>
<td>5264828.67391</td>
<td>277000.85843</td>
<td>-0.0046</td>
<td>-0.0021</td>
</tr>
<tr>
<td>5</td>
<td>5273283.25492</td>
<td>320050.72138</td>
<td>-0.0033</td>
<td>-0.0018</td>
</tr>
<tr>
<td>6</td>
<td>5267793.00436</td>
<td>351788.79179</td>
<td>-0.0040</td>
<td>-0.0035</td>
</tr>
<tr>
<td>7</td>
<td>5242904.40026</td>
<td>385581.98348</td>
<td>-0.0042</td>
<td>-0.0031</td>
</tr>
<tr>
<td>8</td>
<td>5240967.14461</td>
<td>413834.06427</td>
<td>-0.0050</td>
<td>-0.0032</td>
</tr>
<tr>
<td>9</td>
<td>5241104.63290</td>
<td>314071.67861</td>
<td>-0.0045</td>
<td>-0.0027</td>
</tr>
<tr>
<td>10</td>
<td>5233031.94533</td>
<td>314071.45155</td>
<td>-0.0059</td>
<td>-0.0032</td>
</tr>
<tr>
<td>11</td>
<td>5220287.08715</td>
<td>309779.45390</td>
<td>-0.0054</td>
<td>-0.0023</td>
</tr>
<tr>
<td>12</td>
<td>5175772.26759</td>
<td>307157.54510</td>
<td>-0.0063</td>
<td>-0.0024</td>
</tr>
<tr>
<td>13</td>
<td>5196843.37874</td>
<td>262762.71094</td>
<td>-0.0059</td>
<td>-0.0033</td>
</tr>
<tr>
<td>14</td>
<td>5232054.46519</td>
<td>264582.07064</td>
<td>-0.0032</td>
<td>-0.0029</td>
</tr>
</tbody>
</table>
4. CONCLUSION

This paper has illustrated a new method of transformation of conformal coordinates of type Lambert Conic from a global datum to a local datum, and vice versa. The method takes advantage of Taylor expansion of coordinate differences and differences in ellipsoidal form parameters up to second order terms.

It is observed that in order to achieve millimeter precision in the computation of transformed coordinates, the second order terms of the transformation equations should not be neglected.

REFERENCES


Soler, T. (1976): On differential transformation between Cartesian and Curvilinear (geodetic) coordinates, The Ohio State University, Department of Geodetic Science, Report 236, Columbus.

BIOGRAPHICAL NOTES

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Professor E. Grafarend is a Professor emeritus of the Geodetic Institute of the University of Stuttgart, Germany since April 2005. He has over 300 scientific publications and many books. The scientific work of Prof. Grafarend does not only cover most parts of geodesy but has also many connections to mathematics and physics. Professor Grafarend is one of the most outstanding geodesists worldwide.
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2004 – Date: Senior Lecturer, Department of Geoinformatics and Surveying, University of Nigeria, Enugu Campus, Nigeria.
2002 – 2004: Research Fellow/Visiting Scientist, at the Remote Sensing laboratory, JBIDR, Ben Gurion University of the Negev, Sede Boker Campus, Israel.
1998 – 2002: Project Scientist and Consultant with InfoGraph Consultants, (a division of MOSEBETH Nig Ltd), Nigeria
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APPENDIX

Curvilinear Datum Transformation

Points on the Earth surface are described by Cartesian coordinates \(\{x,y,z\}\), \(\{X,Y,Z\}\) of the embedded space assumed to be a Weitzenboeck vector space of three dimensions with respect to the center of mass of the Earth, labeled \(Q\) as the origin and the principal axis of an ellipsoid of revolution \(E_{a,b}^2, E_{A,B}^2\) in a local or global frame of reference. The two sets \(\{x,y,z\}\), \(\{X,Y,Z\}\) are related by a rectangular datum transformation which leaves as a passive transformation angles and distance ratios equivariant (“invariant”). In its linear variant, the seven-parameter transformation which leaves angles and distance ratios equivariant is the conformal group \(C_7(3)\) is parameterised, “closed to the identity”, by three parameters \(\{t_x,t_y,t_z\}\) of translation, three parameters \(\{\alpha, \beta, \gamma\}\) of rotations and one scale parameter \(s\). With the forward and backward transformation close to identity of conformal type, the final form (A1) is gained.

If we perturb local “surface normal” ellipsoidal longitude \(\lambda\), ellipsoidal latitude \(\varphi\), and ellipsoidal height \(h\) by means of (A14), \(\lambda = \Lambda + \partial \Lambda\), \(\varphi = \Phi + \partial \Phi\), \(h = H + \partial H\) we are lead to the Taylor expansion \(\{x(\Lambda + \partial \Lambda, \Phi + \partial \Phi, H + \partial H), y(\Lambda + \partial \Lambda, \Phi + \partial \Phi, H + \partial H), z(\Lambda + \partial \Lambda, \Phi + \partial \Phi, H + \partial H)\}\), namely to (A4). The zero order vector-valued term \(j\) contains the effect of different form parameters \(\{2_e, a^2, e\}\), \(\{2_A, E^2\}\), respectively. In contrast the first order matrix-values term \(J\) represent in a first approximation the influence of the variation in ellipsoidal coordinates \(\{\Lambda, \Phi, H\}\), \(\{\lambda, \varphi, h\}\), respectively.

A first remark is concerned with the functional representation \(h(\lambda, \varphi)\), \(H(\Lambda, \Phi)\), respectively. Since the curvilinear coordinates represent the earth surface conventionally considered to be a two–dimensional Riemann manifold, two parameters/two coordinates, namely \(\{\lambda, \varphi\}\), \(\{\Lambda, \Phi\}\) respectively are sufficient to coordinate points on the earth surface. Accordingly the height \(H, h\) must be functions \(H(\Lambda, \Phi)\), \(h(\lambda, \varphi)\) of the surface parameters/surface coordinates \(\{\Lambda, \Phi\}\), \(\{\lambda, \varphi\}\), respectively. These functions of type orthonormal on \(E_{a,b}^2\) or \(E_{A,B}^2\), respectively, have been numerically given in Grafarend and Engels (1992).

A second remark refers to the zero terms of the Taylor expansion of the function:

\[
\begin{align*}
x(\lambda, \varphi, h; a, e^2) &= x(\Lambda + \partial \Lambda, \Phi + \partial \Phi, H + \partial H; a, e^2) \\
y(\lambda, \varphi, h; a, e^2) &= y(\Lambda + \partial \Lambda, \Phi + \partial \Phi, H + \partial H; a, e^2) \\
z(\lambda, \varphi, h; a, e^2) &= z(\Lambda + \partial \Lambda, \Phi + \partial \Phi, H + \partial H; a, e^2)
\end{align*}
\]

In the previous contributions Grafarend, E., Krumm, F., and F. Okeke (1995), and Grafarend, E. and F. Okeke (1998) we have linearized those functions in terms of \(a = A + \partial A\), \(e^2 = E^2 + \partial E^2\), namely with Jacobi terms \(j_1 \partial A + j_2 \partial E^2\), too! Here, as well as in Grafarend and R. Syffins (1998) we have refrained from such an additional linearization for
numerical reasons. A gain in numerical precision is achieved without linearization, or in order words, linearization in terms of \((\partial A, \partial E^2)\) is not accurate enough.

Let us go back to the “\textit{direct curvilinear datum transformation}” (A6), which is achieved in the following way. The Taylor expansion leads to the Cartesian coordinate increments (A4) of type
\[
\begin{align*}
x - X &= f_\lambda(\lambda - \Lambda, \varphi - \Phi, h - H; a, e^2, A, E^2) \\
y - Y &= f_\varphi(\lambda - \Lambda, \varphi - \Phi, h - H; a, e^2, A, E^2) \\
z - Z &= f_h(\lambda - \Lambda, \varphi - \Phi, h - H; a, e^2, A, E^2),
\end{align*}
\]

namely in the linearized form in terms of \((\lambda - \Lambda, \varphi - \Phi, h - H)\). The solution of the system of equations (A4) is (A5) which relates
\[
\begin{align*}
\lambda - \Lambda &= g_\lambda(x - X, y - Y, z - Z, a, e^2, A, E^2) \\
\varphi - \Phi &= g_\varphi(x - X, y - Y, z - Z, a, e^2, A, E^2) \\
h - H &= g_h(x - X, y - Y, z - Z, a, e^2, A, E^2),
\end{align*}
\]

namely in the linearized form in terms of \((x - X, y - Y, z - Z)\). Finally by means of the “\textit{direct Cartesian datum transformation}” close to the identity, namely (A1), inserted into (A5) as quoted above we arrive at the destination point the “\textit{direct curvilinear datum transformation}” close to the identity, namely (A6). The influence of the difference in form parameters \((a, e^2), (A, E^2)\), respectively is to be seen in the transformed form parameter vector \(K = J^{-1}j\), \(j\) given in equation (A9). In contrast, the 7 parameters of the conformal group \(C_7(3)\) of type translation \((t_x, t_y, t_z)\), rotation \((\alpha, \beta, \gamma)\) and scale \(s\) act on the datum parameter Jacobi matrix \(A = J^{-1}S\), \(J^{-1}\) given in equation (A10), namely based on the explicit form \(j\), \(J^{-1}\) reviewed in equations (A7) and (A8).

The conformal group \(C_7(3)\), forward and backward transformation, Cartesian coordinates \(\{x, y, z\}\) of local type versus Cartesian coordinates \(\{X, Y, Z\}\) of the global type covering \(\mathbb{R}^3\) equipped with an Euclidean matrix is given as follows.

\[
\begin{bmatrix}
x - X \\
y - Y \\
z - Z
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & -Z & Y \\
0 & 1 & 0 & +Z & 0 \\
0 & 0 & 1 & -Y & +Z
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
t_x \\
t_y \\
t_z
\end{bmatrix}
\]

(A1i)
The corresponding Curvilinear backward – forward transformation is then given as follows

\[
\begin{bmatrix}
x - X \\
y - Y \\
z - Z \\
\end{bmatrix} = S \begin{bmatrix}
t_x \\
t_y \\
t_z \\
\end{bmatrix} \quad \forall S \in R^{3\times7} \quad \text{(A1ii)}
\]

Applying Taylor expansion we have

\[
\begin{bmatrix}
\lambda - \Lambda \\
\phi - \Phi \\
h - H \\
\end{bmatrix} = \mathbf{j} + \mathbf{J} \begin{bmatrix}
\lambda - \Lambda \\
\phi - \Phi \\
h - H \\
\end{bmatrix} + O_2(\delta \lambda^2, \delta \phi^2, \delta h^2) \quad \text{(A4)}
\]

\[
\begin{bmatrix}
\lambda - \Lambda \\
\phi - \Phi \\
h - H \\
\end{bmatrix} = -J^{-1} \mathbf{j} + J^{-1} \begin{bmatrix}
x - X \\
y - Y \\
z - Z \\
\end{bmatrix} + O_2(\delta \lambda^2, \delta \phi^2, \delta h^2) \quad \text{(A5)}
\]

The vector \( \mathbf{j} \) and the matrix \( J^{-1} \) are given in equations (A7) and (A8) respectively. The direct curvilinear datum transformation is then given by

\[
\{ \delta \Lambda \} = \begin{bmatrix}
\lambda - \Lambda \\
\phi - \Phi \\
h - H \\
\end{bmatrix} = K + \mathbf{A} \begin{bmatrix}
t_x \\
t_y \\
t_z \\
\end{bmatrix} ; K = -J^{-1} \mathbf{j} \in R^{1\times7} \quad \text{(A6)}
\]
The datum parameter Jacobi matrix, \( J \) is the matrix of partial derivates of \( x, y, z \), with respect to \( j, l, h \). The matrix \( J^{-1} \) is derived from \( J \) and is given as

\[
J^{-1} = \begin{bmatrix}
\sin \Lambda & \cos \Lambda & 0 \\
(n(\Phi) + H) \cos \Phi & (n(\Phi) + H) \cos \Phi & 0 \\
\cos \Lambda \sin \Phi & -\sin \Phi \sin \Lambda & \cos \Phi \\
m(\Phi) + H & m(\Phi) + H & m(\Phi) + H \\
\cos \Phi \cos \Lambda & \cos \Phi \sin \Lambda & \sin \Phi \\
\end{bmatrix}.
\] (A8)

Given that

\[
P(\Phi) = \begin{bmatrix}
a(1 - E^2 \sin^2 \Phi)^{1/2} \\
a(1 - E^2 \sin^2 \Phi)^{1/2} \\
\end{bmatrix}, \quad Q(\Phi) = \begin{bmatrix}
a(1 - e^2)(1 - E^2 \sin^2 \Phi)^{1/2} \\
A(1 - E^2)(1 - e^2 \sin^2 \Phi)^{1/2} \\
\end{bmatrix}.
\]

Then

\[
K := J^{-1} j = \begin{bmatrix}
k1 \\
k2 \\
k3 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
-\sin \Phi \cos \Phi (Pe^2 - E^2) \\
N [P - 1 - \sin^2 \Phi (Pe^2 - E^2)]
\end{bmatrix}.
\] (A9)

\[
A := J^{-1} S = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37}
\end{bmatrix}.
\] (A10i)
\[
a_{11} = -\sin \Lambda / [(PN + H) \cos \Phi ] \\
a_{12} = +\cos \Lambda / [(PN + H) \cos \Phi ] \\
a_{13} = 0 \\
a_{14} = \left[(1 - E^2)PN + H\right] \sin \Phi \cos \Lambda / \left[(N + H) \cos \Phi \right] \\
a_{15} = \left[(1 - E^2)PN + H\right] \sin \Phi \sin \Lambda / \left[(N + H) \cos \Phi \right] \\
a_{16} = -(N + H) / (PN + H) \\
a_{17} = 0
\]

\[
a_{21} = -\sin \Phi \cos \Lambda / (QM + H) \\
a_{22} = -\sin \Phi \sin \Lambda / (QM + H) \\
a_{23} = \cos \Phi / (QM + H) \\
a_{24} = -\left[N^2 (1 - E^2) + MH\right] \sin \Lambda / \left[M (QM + H)\right] \\
a_{25} = \left[N^2 (1 - E^2) + MH\right] \cos \Lambda / \left[M (QM + H)\right] \\
a_{26} = 0 \\
a_{27} = -E^2 N \sin \Phi \cos \Phi / (QM + H)
\]

\[
a_{31} = \cos \Phi \cos \Lambda \\
a_{32} = \sin \Phi \sin \Lambda \\
a_{33} = \sin \Phi \\
a_{34} = -NE^2 \sin \Phi \cos \Phi \sin \Lambda \quad \text{(A10iv)} \\
a_{35} = NE^2 \sin \Phi \cos \Phi \cos \Lambda \\
a_{36} = 0 \\
a_{27} = H + N(1 - E^2 \sin^2 \Phi)
\]