On the Geographic Methods of Eratosthenes of Kyrene

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SUMMARY

In the frame of the rectification of Ptolemy's data in the "Geographike Hyphegesis" it became necessary to study in detail the geographic work of Eratosthenes of Kyrene, who has introduced, according to Strabon, mathematics and physics into geography.
It turned out that an answer to four questions are of major importance to understand Eratosthenes' work as a natural scientist as well as the ancient development of the natural sciences, of geodesy, astronomy and geography:

1. What definition 1 stade = 600 foot has Eratosthenes used?
2. What can be said about the ancient use of the trigonometric functions?
3. What kind of map design has Eratosthenes used?
4. What was the ancient fate of the heliocentric hypothesis?

An answer to the first three questions is given in this treatise.

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1 INTRODUCTION

Most enigmatic and therefore fascinating is the role Eratosthenes of Kyrene has played in the development of the ancient natural sciences, of geodesy, astronomy and geography. Four questions have to be answered in this respect.

1. A professional construct of a map for the Oikumene using the survey distance data of bematists/navigators required a careful determination of the circumference of the Earth as determined by Eratosthenes to be 252000 stades.
What definition 1 stade = 600 foot has Eratosthenes used?
2. The Thales-triangle, a rectangular triangle in a half-circle with diameter d=1, provides a geometric definition of all trigonometric functions.
What can be said about the ancient use of the trigonometric functions?
3. Eratosthenes mainly established his map of the Oikumene, very famous in ancient times, in the east from the survey data of the Makedonian army of Alexandros the Great (delivered to Alexandria by Patrokles), in the west from the data of Timosthenes and Pytheas.
What kind of map design has Eratosthenes used e.g. to check those survey data for gross errors and inconsistencies?
4. The heliocentric hypothesis of the planetary system was developed already in antiquity by Aristarchos of Samos, a predecessor of Eratosthenes in Alexandria.
What was the ancient fate of the heliocentric hypothesis?

On the one hand for all those four questions a definite answer could not be given until today. On the other hand without an answer to those questions any modern ideas about the development of the natural sciences in antiquity will remain questionable and therefore just ideas.

A definite answer to the first question is given by the research in the last two decades concerning the metric length of the about 40 ancient cubit/ foot-units as provided by archaeological evidence as well as by two literary statements.

Heron tells us that the Egyptian schoinos was divided into 30 stades I, Pliny tells us that the schoinos was divided into 32 stades II as well as 40 stades III, the last one used by Eratosthenes. According to this ancient information we get the condition equation:
1 schoinos = 30 stades I = 32 stades II = 40 stades III.

By taking into account all ancient foot-units known from archaeological evidence there is just one and only one solution left, namely
1 schoinos = 12000 pechys histonikos (Eg. royal cubit) = $12000 \cdot 0.5291$m = 6349m

1 stade I	= 600 pous Ptolemaikos	= $600 \cdot 0.3527$m	= 211.6m
1 stade II	= 600 pous Philetairikos	= $600 \cdot 0.3307$m	= 198.4m

1 stade III = 600 Gudea foot = $600 \cdot 0.26455$m = 158.73m
According to this condition equation no question is left anymore what kind of stade definition Eratosthenes has used to provide the
circumference of the Earth = 252000 stade Eratosthenes = 40000km
and consequently the
diameter of the Earth = $252000/\pi$ = 80200 stade Eratosthenes = 12730km.
How could Erathostenes achieve such an excellent result? About the method used by his predecessor to determine the circumference of the Earth provides Ptolemy the necessary information in book I.5 of the Geographike Hyphegesis (Knobloch et. al. 2003).

Astronomical methods were required in antiquity as well as in our time for **time determination, navigation and geography**; all three subjects are of high practical importance for military and governmental purposes, today as well as in the ancient empires.

Indeed, at the life time of Eratosthenes a new calendar was introduced in Egypt; by introducing a leap-day every four years the length of the year was fixed to 365.25 days. Later on Julius Caesar took over this calendaric concept for the Roman empire.

The use of astronomic methods for navigation and geography is closely connected to the questions 2 and 3. In the sequel an attempt is made to provide an answer to those questions taking into account the very sparse literary information about those subjects with due care.

Last but not least a reader amazed about the high accuracy Eratosthenes had already obtained for his estimation of the circumference of the Earth should be much more amazed about the accuracy of his estimation of the "Astronomical Unit", that is the distance between the Earth and the sun. According to a careful investigation of the literary information handed down to us (Kleineberg 2008) it was the value

1 AU = 804 000 000 stade = 10 050 Earth diameter = 128 million km ~ 150 million km.
And, taking into account the angular diameter of the sun as observed e.g. by Archimedes of Syracuse, he got in addition

diameter of the sun = 4 000 000 stade = 100 diameter of the Earth.
Could it be that the huge sun rotates once a day around the tiny Earth? Hard to belive. According to his careful estimation of the "Astronomical Unit" Eratosthenes may have been called "Beta" in Alexandria, the second adherent of the heliocentric hypothesis of Aristarchos of Samos. The ancient fate of the heliocentric hypothesis is certainly a question requiring further research; an answer will be given in another paper.

A last remark may be opportune. The modern literature is full of funny ideas about the state of the natural sciences in antiquity. Those funny ideas can only come up if somebody does not study very carefully the ancient methods to get precise measurement data as well as the ancient mathematical methods to use those measurement data for the problems of the natural sciences.

2 GONIOMETRY AND TRIGONOMETRIC FUNCTIONS IN ANTIQUITY

As all ancient authors agree upon Greek mathematics started with Thales of Milet (~625 – 547 bc) and his younger colleague Anaximander. It have indeed been very practical problems both scientists were interested in and which required the development of new mathematical methods obviously unknown in Egypt and Babylonia.

"Surveys through the air" were introduced by them, requiring the **definition of directions** and the **measurement of angles** to determine e.g.

- the length of the seasons,
- the height of the pyramids and obelisks,
- the distance of an island or a ship far off-shore in the sea etc.

In the last case striking must have been for both that a ship/ island very far away in the sea could only be seen if those measurements were performed at a hill instead at the shore. Since this takes place in all directions just one reasonable answer was left: the ocean surface must form a sphere.

Indeed, according to old reports noted down by Diogenes Laertios (2./ 3. cent. AD) Anaximander has talked about a spherical Earth in the middle of the universe and not about a cylindrical one as some have believed.

According to Eratosthenes Anaximander was the first one who had established a map of the Oikumene. If he has mapped for this purpose in a first step the meridians on a cylinder and then the cylinder into a plane the strange idea could easily have been arised that he considered the Earth to be a cylindrical body as other reports tell us.

In any case, as we know again from Eratosthenes, Eudoxos of Knidos (~408 – 355 bc) also drew a map of the Oikumene and at his time it was generally accepted that the Earth forms a sphere. The mapping of the surface of a sphere into a plane appropriate for geographic purposes was certainly one problem the Greek mathematicians have been confronted with at least since Eudoxos.

Already Anaximander used a scientific sunclock, a so-called Horologion or Skiotherikos Gnomon/ Skiotheron (shadow intercepting gnomon; see Lelgemann et. al. 2005) to observe e.g. the length of the seasons as well as the true local solar time, called very early by the Greeks "horai isemerinai".

Introducing directions and angles for measurement purposes one needs reference directions as well as angular units. Regarding the first task the Greeks aligned the horizontal circle to the south direction of the meridian; for geodetic purposes the main problem was then how to realise the meridian direction in situ in the field.

Furthermore, they divided the horizontal circle into $2 \cdot 6 = 12$ parts; any of those 12 main horizontal directions got the name of a wind.

Starting from the basic unit
$\alpha = 360°/12 = 30° =$ zodiacal signs
bisection leads to
$\alpha = 15° = \beta\alpha\vartheta\mu o\iota$ (steps) or $\varepsilon\chi\tau\eta\mu o\rho\iota o\iota$ (sixths (of 90°))
$\alpha = 7\,{}^{1}\!/_{2}° = \mu\varepsilon\rho o\iota$ (parts)

Trisecting the last unit one gets the ancient “angular cubit” to 30 daktylos, still used (beside the degree unit) by Hipparch to describe geographical latitudes in the northern part of the Black Sea (Jones 1949, p.283):
$\alpha = 2\,{}^{1}\!/_{2}° = 1\,cubit = 30\,daktylos$
$\alpha = 5' = 1\,daktylos$ (finger width).

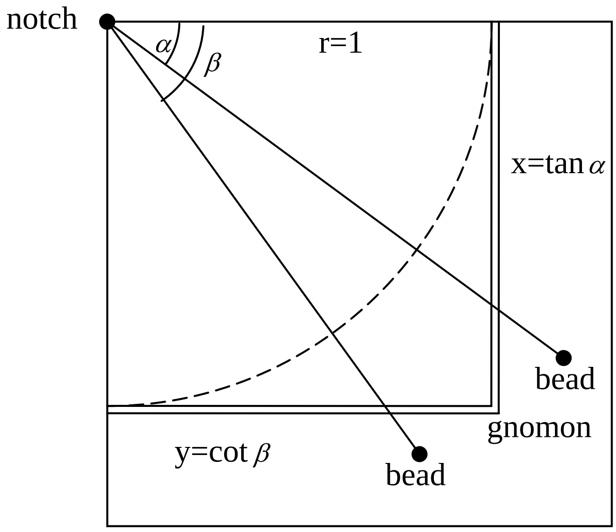


Fig 1: A so-called shadow-square (Horologion)

How can it be explained that the Greeks used the same name for length - as well as for angular - units? Was it because of their measurement technique to get angles?
Of course, a very precise division of an actual solid circle for measurement purposes is not easy and required e.g. trisection of an angle; much easier is a precise division of a solid ruler. Both can be used to measure angles (see fig. 1), but the ruler (in form of a gnomon as defined by Aristoteles) only by an application of trigonometric functions, namely the tangens function.

On the other hand all trigonometric functions are graphically defined very simply by a Thales-triangle (see fig. 2).

With a few exceptions ($\sin 0° = \cos 90° = \tan 0° = \cot 90° = 0$, $\cos 0° = \sin 90° = \tan 45° = \cot 45° = 1$, $\sin 30° = \cos 60° = 1/2$) all trigonometric functions are irrational numbers. The Greek did not consider irrational numbers as numbers at all; just integer and rational numbers have been used by them in arithmetics.

Irrational numbers have probably been treated in books about logistic as well as interpolation methods between a series of data. There have been ancient books about logistic, but unfortunately none has been handed down to us.

Natural scientists such as Archimedes of Syracuse confined irrational numbers like π or the angular diameter of the sun δ by two limits defined by rational numbers, e.g.

$$3.14085 \sim 3^{10}\!/_{71} < \pi < 3^{10}\!/_{70} \sim 3.14286$$

$$27' = {}^{90°}\!/_{200} < \delta < {}^{90°}\!/_{164} \sim 33'$$

The loss of ancient books about logistic seem to be the reason why our knowledge obtained from the ancient literature about the use of trigonometric functions by the Greeks, very important for the natural sciences such as geography, is extremely sparse and an open question until today.

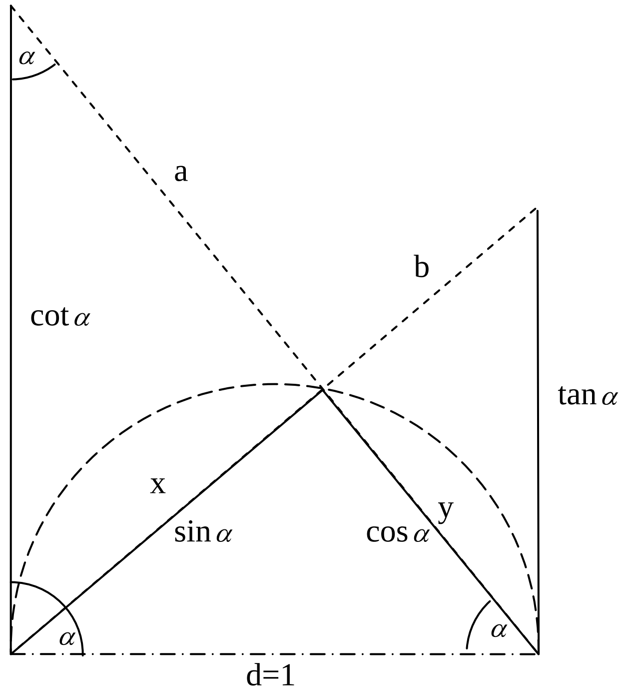


Fig 2: Thales-triangle and the trigonometric functions

Of course, all basic relations between trigonometric functions can immediately be recognised at the Thales-triangle, for example

$$\sin^2\alpha + \cos^2\alpha = 1,$$

$$\tan\alpha : 1 = \sin\alpha : \cos\alpha,$$

$$\cot\alpha : 1 = \cos\alpha : \sin\alpha,$$

$$\tan\alpha \cdot \cot\alpha = 1,$$

Moreover, one can immediately set up the important double proportion (see fig. 2)

$$\frac{a}{x}=\frac{x}{y}=\frac{y}{b}=\frac{\cos\alpha}{\sin\alpha}=\cot\alpha\ .$$

According to Eratosthenes it was Hippokrates of Chios, a contemporary of Platon (429 – 348 bc), who has found out that a solution for this double proportion must be sought in order to double the volume of a cube $a^2 = 2b^3$. One of the many ancient solutions for this problem, attributed to Platon, used a (mobile) construct as shown by fig. 2.

A main step in goniometry was the invention of an ingenious curve by the sophist Hippias of Elis, a contemporary of Sokrates, later on called "quadratix". Using Cartesian coordinates this curve can be described by

$$x = (1-\alpha°/90°)a \qquad y = x\tan\alpha\ .$$

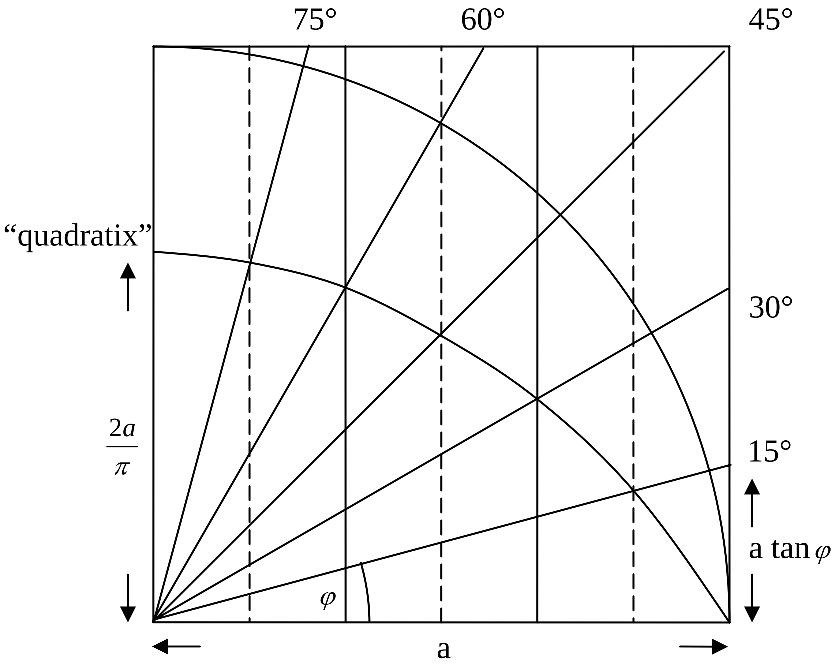


Fig 3: The „quadratix" of Hipparch of Elis: a graphic table for the tangens-function

For $\alpha = 90°$ one gets x=0 and $\lim y/a = 0\cdot\tan\alpha = 0\cdot\infty = 2/\pi$; this limit could therefore be used for the quadrature of a circle as the Geeks have already recognised.

Bisecting the three zodiacal signs ($3\cdot 30° = 90°$) several times the "quadratix" can be established easily by a geometric construction (see fig. 3)

a=7,5	α	15°	30°	45°	60°	75°	90°
	x	6 1/4	5	3 3/4	2 1/2	1 1/4	0
	y	1.67	2.89	3 3/4	4.33	4.67	$a(2/\pi)$ =4.77

Table 1: Cartesian coordinates of the „quadratix"

Having constructed the curve one can proceed as follows. Given an angle α , one computes first x, takes y from the curve and gets $\tan\alpha = (x/y)$. Given $\tan\alpha$, one takes y from the curve, computes x from $y/\tan\alpha$ and gets α from $\alpha = 90(1-(x/\alpha))$. With other words the "quadratix" represents a geometric construct for a table of tangens-functions.

At the time of Archimedes and Aristarchos, trigonometric functions such as $\sin\alpha$ and $\tan\alpha$ have been a well-known concept for the Greeks. For example, both scientists used without any explaining comment for the reader the inequality (see Neugebauer 1975, p.772)

$$\frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}, \quad \alpha < \beta$$

where α and β are given in radian: $\alpha_r = \alpha°(\pi/180°)$.

Goniometry was certainly advanced further by Archimedes (see appendix). Biruni has preserved a "Lemma of Archimedes" (see Neugebauer 1975, p.776), extremely useful to derive geometrically goniometric formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Moreover, Archimedes derived also the goniometric relation $\sin^2(\alpha/2) = (1-\cos x)/2$.

Eudoxos (as probably Anaximander before him) was confronted with the problem how to design a map of the Oikumene. As Strabon and later on Ptolemy have told us the most obvious solution for this problem was the construction of a concrete sphere (of a diameter d=6 cubit~3m; scale α =1:4 million) and draw the map on this sphere, using meridians and parallel circles as reference grid.

The distance measurements of the bematists/ navigators had to be separated then into two components: a north-south component called by the Roman "cardo" and an east-west component called by the Romans "decumanus".

Of course, using the distance data of the bematists/ navigators to construct such a map on the sphere one needs an estimation of the circumference of the Earth. The "mathematicians" at the time of Eudoxos had determined already the circumference of the Earth to be 400000 stade. Unfortunately, there is no information about the definition of the stade those mathematicians have used.

At least already Eudoxos of Knidos (if not Anaximander before him) must have recognised that the "decumanus" $p(\phi)$ between two meridians depends on the latitude ϕ according to

$$p(\phi) = \Delta\Lambda\cos\phi \, .$$

If $d(\phi)$ is measured and ϕ is given one gets easily the longitude difference $\Delta\Lambda$ from

$$\Delta\Lambda = p(\phi)/\cos\phi \, .$$

Indeed, we know from Ptolemy that the cosine-function has already been used by older geographers. At the end of the "Geographike Hyphegesis", book I.20 he just remarks without any further comment:

For in such units as the equator is 115

- *the parallel 36° from the equator and drawn to Rhodes is 93* [~115cos36°=93.04]
- *and the parallel 63° and drawn through Thule is 52* [~115cos63°=52.21]

Why have the older geographers used the mysterious factor 115? There are two explanations. Later on in astronomy Hipparch used a table of half-chords – in India those tables were called "kadarka" – or sine values multiplied by the factor 3438'=57.3°~180°/π. Was 115 just an approximation of 360/π=114.6 and the equator measured in units of π=3 1/7?
On the other hand the number 115 is the basic parameter for an ingenious kind of equidistant mapping design of the northern half- sphere, extremely expedient if distance data are used to establish a map for the Oikumene.

3 ON THE DESIGN OF GEOGRAPHIC MAPS IN ANTIQUITY

Nearly nothing is known about this subject from literature with one exception: the Geographike Hyphegesis of Claudius Ptolemy.

Strabon emphasised the use of a concrete globe for a map of the Oikumene using as a reference grid meridians and parallel circles. For a planar map both should be drawn rectangular to each other, but he mentioned also the possibility to tilt the meridians.

Indeed, the only professional report about possibilities for a geographic map design is the report given by Ptolemy (see Berggren and Jones 2000). Since his report is somewhat confusing (having therefore led to funny modern interpretations) the information given in this section should be considered as comments from a modern expert for geographic map design. Ptolemy remarks (similar as Strabon):

This undertaking [to drew a map of the Oikumene] *can take two forms.*

- *The first sets out the Oikumene on a part of a spherical surface*
- *and the second on a plane.*

Each of the two approaches is characterised in the following way.

Making the map on the globe one gets directly the likeness of the earth's shape, and it does not call for any additional device to achieve this effect; but it does not conveniently allow a size capable of containing most of the things that have to be inscribed on it.

Drawing the map on a plane eliminates these [difficulties] *completely; but it does require some* [special] *method to achieve a resemblance to a picture of a globe such, that on the flattened surface, too, the intervals* [cardo/ decumanus] *established on it will be in as good proportion as possible to the true* [ones].

With other words, an **equidistant** design for the planar map is required. Let us see how Ptolemy did achieve this in section 24: *Method of making a map of the Oikumene in the plane in proper proportionality with its configuration on the globe.*

The basic concept of the mapping design (it is **not** a projection onto a cone such as the stereographic projection onto a tangential plane) described in the Geographike Hyphegesis can easily be grasped from fig. 4.

One draws the parallels as circles around a point A and straight lines through A such that the circular distance for $\Delta\Lambda$ at the equator circle have the same scale as the distance for $\Delta\Phi$ at the central meridian. The distance R of the point A from the equator can be arbitrarily chosen.

We may choose R according to the condition that at the parallel through Thule at $\Phi_T = 63°$ the scale is preserved, too. Since we have $p_s = \Delta\Lambda \cos\Phi$ at the sphere and $p_p = \Delta\Lambda(1-\Phi/R)$ at the plane we get for R as a condition equation $\cos\Phi_T = 1-\Phi_T/R$ or

$$R = \Phi_T/(1-\cos\Phi_T) .$$

For a few values of Φ_T the corresponding distances R are given in table 2.

Φ in degrees	63°	63°15'	63°25'
R	115.4	115.0	114.8

Table 2: R for various degrees of Φ_T

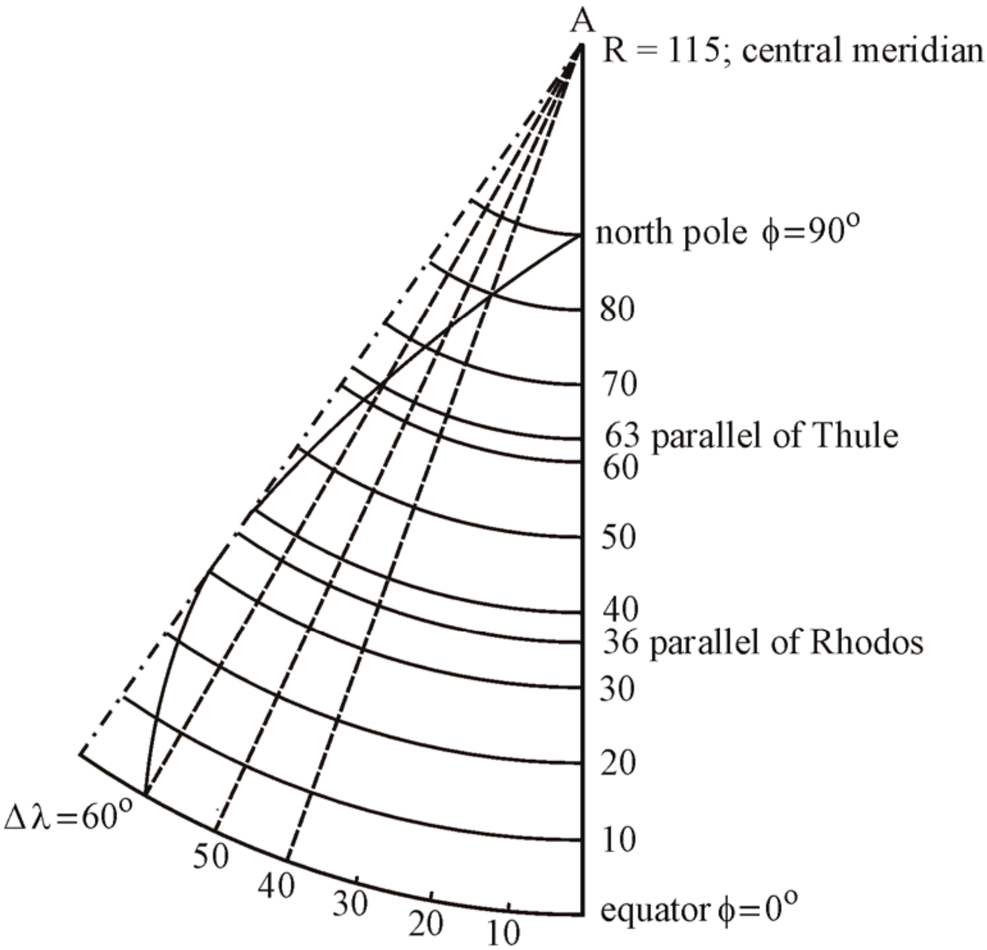


Fig 4: Western part of the reference meridian: (nearly) equidistant map design

Regarding the meridians at the remaining latitudes Φ one gets as differences δp between $p_s = \Delta\Lambda \cos\Phi$ at the sphere and $p_p = \Delta\Lambda(1-\Phi/R)$ at the plane

$$\delta p = p_s - p_p = \Delta\Lambda(\cos\Phi - (1-\Phi/R)) = \Delta\Lambda(\cos\Phi + \Phi/R - 1) = (\Delta\Lambda/115)(115\cos\Phi - (115-\Phi)) .$$

For $\Delta\Lambda$ =60° the correction terms δp are listed in table 3. Using those reduction terms the meridian curve (solid line) can easily be drawn as done in fig. 4 for the meridian $\Delta\Lambda$ =60° as an example.

Φ	0	10	20	30	31.5	40	50	60	63	70	80	90
δp	0	4.3	6.8	7.6	7.6	6.8	4.7	1.3	0.1	-3.0	-7.8	-13.0

Table 3: Correction terms δp for the meridian $\Delta\Lambda$ =60°

Of course, an exact equidistant mapping of the sphere into the plane is impossible. But in the ancient map design equidistancy is preserved

- along all parallel circles (decumanus)
- and along the straight lines forming the reference for the curved meridian lines (cardo).

Indeed, this is the most expedient kind of mapping design if precise bematists/ navigator-data should be checked for gross-errors and inconsistencies.
The mapping design as described so far is usually called in modern times “Ptolemy’s second projection”.

In contrast to what Ptolemy says at the end of his description it is of course possible to draw the map using just a revolving ruler and a circle for the equator. If for a location Φ and $\Delta\Lambda$ to the reference meridian are given one simply has to direct the ruler to a point $\Delta\Lambda + \delta p$ at the equator circle. Did Ptolemy have grasped this? Hard to believe.

Ptolemy introduced also a “first projection”, trying to avoid curved meridians. In book I.21 he had stated before:

For this reason it would be well to keep the lines representing the meridians straight, but those that represent the parallels as circular segments about one and the same centre, from which one will have to draw the meridian lines.

Since it is impossible to preserve for all the parallels their proportionality on the sphere, it would be adequate

- *to keep this for the parallel through Thule and the equator*
- *and to devise the parallel that is to be drawn through Rhodes in proportion to the meridian as Marinos does, that is in the approximate ratio of similar arc of 5:4* [=1.2~1/cos36°]

Obviously guided by a similar proceeding of Marinos Ptolemy simply fixed the straight lines for the longitudes at the parallel-circle of Rhodes instead at the equator (as shown in the figure by the line −·−·−·−·−). Did he really understand that he changed then at the same time the scale at the parallel of Thule and at the equator by n=cos36°/(1-36/115)=1.2? Again hard to believe according to what he wrote in I.21.

The modern assumption that Ptolemy has invented in particular the second map design is just a modern idea; there is not any information to support it. It must have been indeed a very qualified mathematician, such as Eudoxos or Eratosthenes, who has invented such an ingenious mapping design and Ptolemy may have found a description of it in the library of Alexandria. His own description of this simple and ingenious method, nevertheless, can only be judged as fussy from an expert point of view. Indeed, it may even support the assumption that Ptolemy never has drawn a geographic map at all (see Berggren and Jones 2000, p.46, for a further discussion about this question).

4 SOME REMARKS ABOUT ERATOSTHENES' MAP OF THE OIKUMENE

Very famous in ancient times was the map of Eratosthenes for the Oikumene. Since Ptolemy does not say anything about the work of his famous predecessor in Alexandria (he does not even mention his name in the Geographike Hyphegesis) our only source of information about this work is Strabo. In order to reconstruct at least the main features of the map of Eratosthenes one faces several difficulties.

a) As becomes obvious from their critical comments about it neither Strabo nor Hipparch have ever seen the map of Eratosthenes.
b) Strabo mention that he has "corrected" specifications of Eratosthenes without to mention it in case those seemed to him obviously wrong (section 4.1).
c) The question must be answered how Eratosthenes has defined his reference meridian through the Canobic mouth of the Nil (section 4.2)
d) The question must be answered how Eratosthenes has defined the difference between two meridians (section 4.3).

4.1 Latitudes according to Eratosthenes

The reader may keep in mind that the Greeks, beside the determination of the Cardo, had three methods to measure geographic latitudes by pure astronomical observations.

1. For $\Phi < 24°$ observation of that day when the sun was in the zenith, determining the declination=latitude from the ecliptical longitudes of the sun (Philo-method).
2. Observation of the zenith distance z of the sun with a Skiotherikos Gnomon getting $\Phi = z + \delta$ (Pytheas method).
3. Observation of the length of the longest day at the summer solstitium (Hipparch-method).

Whereas the first two methods may provide the latitude Φ with an accuracy of about 5'-10', the third method will provide (without certain corrections unknown to the Greeks) a value about 2° too large.

Since Strabo was obviously sure that Pytheas had placed Thule under a latitude of about Φ =90°-24°50'=66°10' (modern arctic circle) it seems to be that he had "corrected" (without to mention this) Eratosthenes' latitude data north of Rhodes in order to agree with that assumption (Jones 1949, p.233). Substantiated modifications for the data at the northern limit of Taurus range and north of it are given therefore in table 4, too.

The modifications are based on the following information:

1. For the latitude differences between Rhodes/ Lysimachia lying at the southern/ northern border of the Taurus range the ancient value $\Delta\Phi$ =3000 stades for the Taurus width was used.
2. For the latitude difference between Lysimachia/ Byzantium the modern value of $\Delta\varphi$ =27'=300 stades has been used.
3. For the latitude difference between Byzantium/ Borysthenes the ancient value $\Delta\Phi$ =3800 stades provided by Hipparch was used.
4. For the latitude difference between Borysthenes/ Thule the ancient value $\Delta\Phi$ =11500 stades provided by Eratosthenes has been used.

The latitude value Φ =63° for Thule, introduced by Marinos/ Ptolemy without any further comment as northern border of the Oikumene, agrees nearly exactly with the (modified) value given by Eratosthenes for this northern most point .

location	modern	Strabon		modification		$\delta\varphi - \Phi$
	φ	Φ (stades)	Φ	Φ (stades)	Φ	
Meroe/ Bagrawia	17° 00’	11800	16° 24‘			+10’
Syene/ Aswan	24° 05‘	16800	24° 00‘			+5’
Alexandria	31° 15‘	21800	31° 10‘			+5’
Rhodes	36° 25‘	(25550)	36° 30‘			-5’
Athens	38° 00‘	(26800)	38° 15‘			-15’
Hellespont/ Lysimachia	40° 35‘	29900	42° 45‘	28550	40° 45’	-10’
Byzantium/ Istanbul	41° 02‘	-	-	28850	41° 13’	-10’
Borystenes/ Ochakov	46° 35‘	34900	49° 50‘	32650	46° 40’	-5’
Thule/ Isl. Smola	63° 25‘	46400	66° 15‘	44150	63° 05’	-20’

Table 4: Latitude data according to Eratosthenes

4.2 The reference meridian by Eratosthenes

According to Strabon Eratosthenes has chosen the meridian through the Canobic mouth of the Nil as reference meridian, running through the two Cyanean islands in the Pontus (Black Sea). Where have been those Cyanean islands? Strabo (as well as Ptolemy) located those at the outlet of the Bosporus into the Pontus, but one can not find any island there. According to Hipparch, on the other hand (Jones 1949, p.351), the Cyanean islands have been 5600 stades distant to the west from Phasis (Poti in Georgia) and there are indeed islands, namely the Kefken Adasi (s=11° 25’ cos41.5°$\cdot$700= 6000 stades).

Strabo reports also that this meridian runs through the Chalidonia islands near the Gelidonya Burun.

Moreover, Eratosthenes has said that the distance between the reference meridian and the meridian of Thapsakus at the Euphrat is 6300 stades. A preliminary rectification of Ptolemy’s data in the Geographike Hyphegesis has shown that Thapsakus has to be identified with the modern town As Sabkhah near Nicephorium (Ar Raqqah). The longitude difference between Rashid in Egypt and As Sabkhah in Syria is indeed $\Delta\lambda$ =8°52’=6200 stades Eratosthenes.

Strabon	Modern name	φ	λ
Canobic mouth	At Rashid	31° 25’	30° 25’
Chelidonia isl.	At Gelidonya Burun	36° 14’	30° 25’
Cyanean isl.	Kefken Adasi	41° 14’	30° 15’
Phasis	Poti	42° 09’	41° 40’
Thapsakus	As Sabkhah	35° 48’	39° 17’

Table 5: Location at the reference meridian of Eratosthenes

4.3 Longitudes according to Eratosthenes

The reader may keep in mind that a precise determination of longitude differences by pure astronomical methods requires in situ the knowledge of the precise local solar time at the reference meridian, provided today by a chronometer. In antiquity only the measurement of the decumanus by bematists/ navigators could be used to determine longitude differences $\Delta\Lambda$.

Strabo complained that the longitude data provided by Eratosthenes have been completely wrong in particular for the part of the Oikumene north of the Taurus range and for the east-west distance between the Canobus mouth of the Nile and Carthage. He as well as Hipparch have obviously not realised that the length data provided by Eratosthenes have been longitude differences measured at the equator, as can easily be seen by the comparison with modern data in table 6.

Indeed, Eratosthenes data agree very well with the longitude differences of those main locations along the southern border of the Taurus range with the exception Carthage/ Gibraltar.

But one can not exclude an explanation that Eratosthenes has provided for this special case also the parallel distance at the southern limit of the Taurus range and Strabo has reported this value. Regarding the proportion 5:4=1.2 the longitude difference will be $\Delta\Lambda$ =10000 as used in the table.

It is remarkable how careful Eratosthenes had determined the longitude extension of the Oikumene. For an analysis of the other data provided by Eratosthenes one has to wait for a rectification of Ptolemy’s data in the Geographike Hyphegesis.

Strabo	Modern location	φ	λ	$\Delta\lambda$		Eratosthenes
Mouth of the Indus	at Mt. Daspar	36° 35’	73° 24’			
				21° 16’	14900	14000
Caspian gate	at Qolleh-ye Damavand	35° 56’	52° 08’			
				12° 51’	9000	10000
Thapsakus	As Sabkhah	35° 48’	39° 17’			
				6° 59’	4900	5000
Peluric mouth	at Port Said	-	32° 18’			
				1° 53’	1300	1300
Kanobic mouth	at Rashiel	-	30° 25’			
				20° 09’	14100	13500
Karchedon	Carthage	36° 54’	10° 16’			
				15° 37’	10900	(10000)
Pillars of Herakles	at Gibraltar	36° 09’	-5° 21’			
				3° 35’	2500	3000
Sacred promontory	at Sagres	37° 01’	-8° 56’			

Table 6: Longitude data provided by Eratosthenes

5 FINAL REMARKS

In the course of a rectification of the geographic data given in the Geographike Hyphegesis it turned out that Ptolemy, without saying this explicitly, has used older information. For example, his (wrong) latitudes for Byzantium and Carthage can already be found by Strabo. For any professional analysis of the main distortions in Ptolemy's data set it will therefore be necessary to investigate carefully the method as well as results of his predecessors, in particular the methods.

The in situ establishment of the east-west direction and with this the meridian direction was the fundamental problem for all bematists/ navigators. Sun rise and sun setting happens exactly in east-west direction just two times in the year at the equinoctium.

Was it possible to calculate the horizontal angle α between the exact east-west direction and the sun rising at any time by an astronomical method? Could the very careful determination of the length of the seasons by Kallippos at about 333bc be used as a tool to solve this problem?

	spring	summer	fall	winter
Kallippos	94	92	89	90
Modern data	94.1	92.3	88.7	90.2

Table 7: Length D of the seasons measured by Kallippos at ~333bc

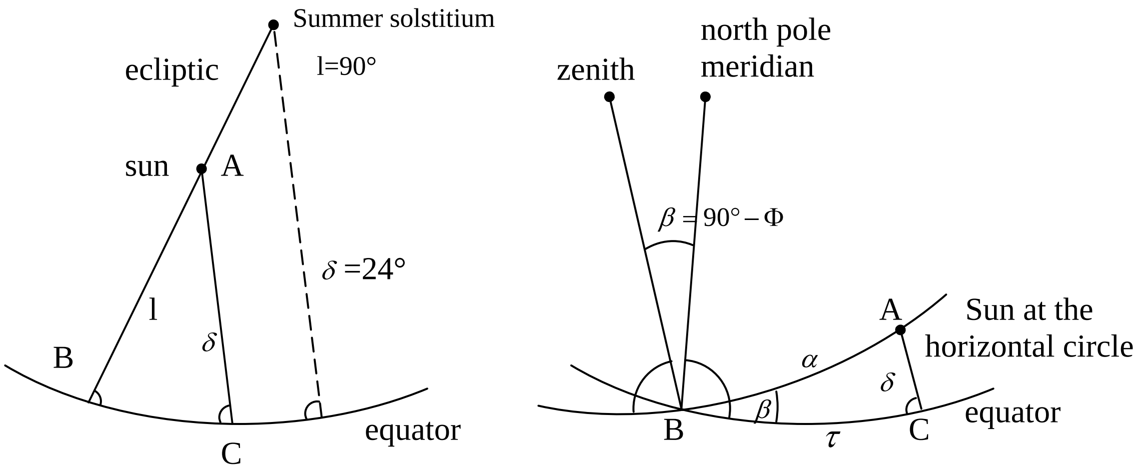


Fig 5: Spherical triangle fort he sun or a star-rise

If one knows for any day of the year the number d of days with respect to the solstitium or equinoctium, respectively, the ecliptical longitude l of the sun can easily be obtained by simple interpolation to be $\Delta l = (d / D) \cdot 90°$.

In a first step one gets then the declination δ of the sun, using ε =24°, from

$$\sin\delta = \sin\varepsilon \sin l \sim (2/5)\sin l$$

In a second step one can compute the azimuth α from

$$\sin\alpha = \sin\delta / \sin(90° - \Phi)$$

and the length of the day $T = 2(6+\tau/15)$ hours from

$$\sin\tau = \tan\delta\tan\Phi .$$

Let us consider for example a location at the parallel of Rhodes Φ =36°. At the summer solstitium the azimuth α of sunrise will be

$$\alpha = \sin^{-1}(\sin 24°/\sin 54°) = 30°10', \; a_N = 90-30 = 60°$$

and the length of the day will be

$$\tau = \sin^{-1}(\tan 24° \tan 36°) = 18.9°, \; T = 2(6+1.25) = 14 1/2$$ hour.

At the winter solstitium the azimuth α of sunrise will be

$$\alpha = \sin^{-1}(\sin(-24°)/\sin 54°) = -30°10', \; a_N = 90+30 = 120°$$

and the length of the day will be

$$\tau = \sin^{-1}(\tan(-24°)\tan 54°) = -18.9°, \; T = 2(6-1.25) = 9 1/2$$ hour.

In particular the azimuth of the sun (or of bright stars near the equator) has been of major importance for bematists/ navigators, as Eudoxos, Kallippos and Aristoteles certainly have recognised. It remains to find out how the Greeks have treated the mathematical aspect of this problem before the invention of spherical trigonometry by Menelaos (1. cent. bc), used by Ptolemy as well as by us for this purpose.

The applications of mathematics for geodetic/ geographic purposes was certainly, besides the introduction of logical concepts in form of proofs, the main motive for the great progress of ancient mathematics by the Greeks. This progress will remain unintelligible if this fact will not be recognised.

Appendix: Archimedes about goniometric relations

Despite the fact that the Thales-triangle provides a clear graphic definition of the trigonometric functions and Archimedes, Aristarchos and Hipparch have used the sine function it is a curious modern idea that the concept of half-chords or sine was introduced by the Indians. Indeed, the sine-table Ptolemy has given in the Almagest is based on the concept of chords but in fact it is a sine-table in the modern sense as the reader may convince himself by proving it (see tab. A1).

α °	½°	1°	1 ½°	2°	2 ½
$\sin\alpha$	0.008727	0.017452	0.026177	0.034899	…
Ptolemy	0^p;31,25	0^p;02,50	0^p;34,14	2^p;05,38	…

Table A1: Chord-table of Ptolemy in the Almagest

The curious modern idea may have come up, because Ptolemy made his astronomical computations without need very fussy and lengthy when he used the data of his sine-table. But why was the concept of chords introduced at all by the Greeks?

Indeed, the basic concept underlying the derivations of goniometric relations by Archimedes is based on the chord-theorem. A chord s in a circle is seen from any points A, B, C etc. at the periphery under the same angle α (see fig. A1).

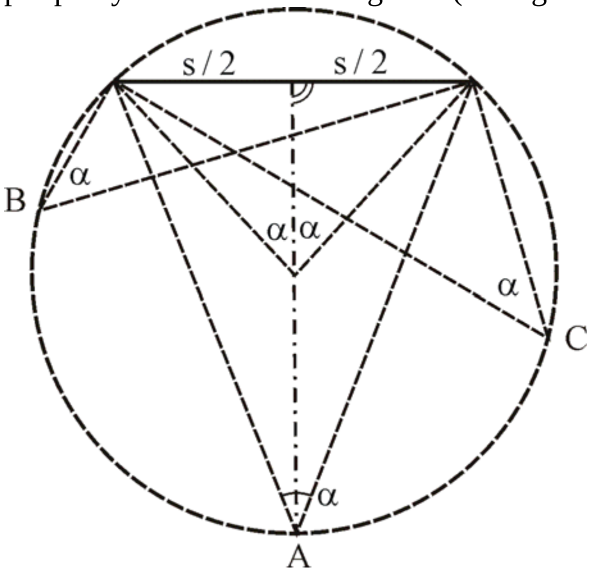


Fig A1: Periphery angles $\alpha = \text{const}$ of a chord in a circle

Archimedes used this property for a derivation of the addition theorems of the sine-function

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

as shown in the following.

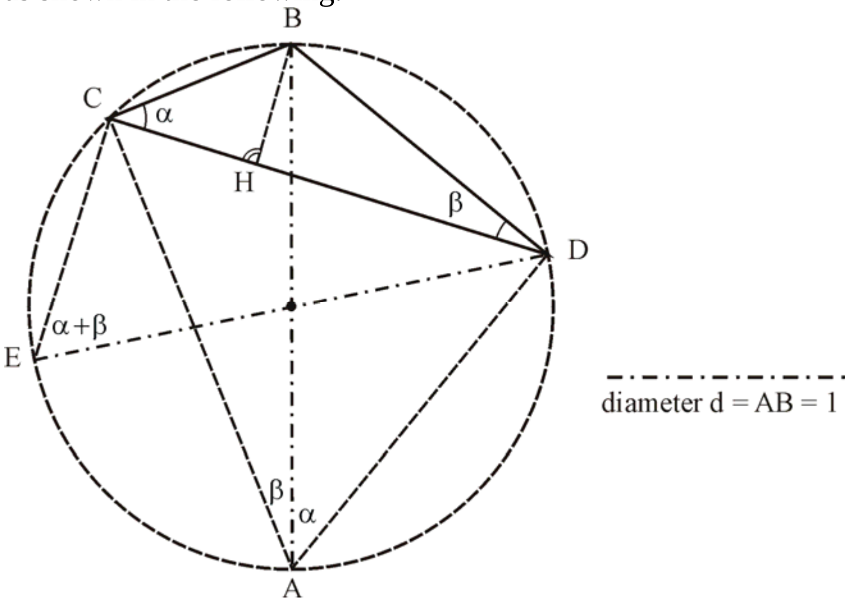


Fig A2a: Basic construction to derive $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Regarding the three Thales-triangle ABC, ABD and GDC in a circle with diameter d=1 one gets $\text{BC}=\sin\alpha$, $\text{BD}=\sin\beta$, $\text{CD}=\sin(\alpha+\beta)$.

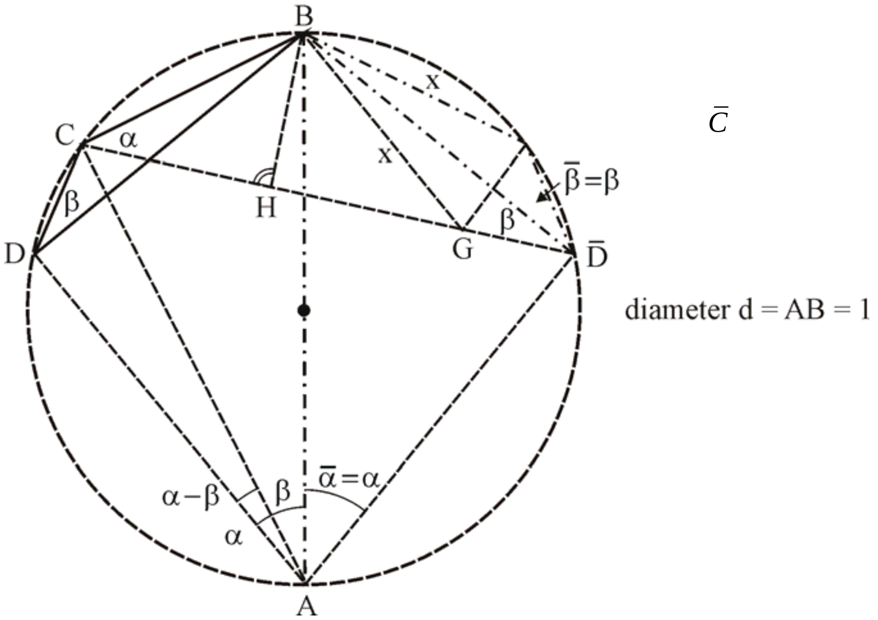
Furthermore, we get $\text{CH}=\text{BC}\cdot\cos\alpha = \cos\alpha\sin\beta$ and $\text{HD}=\text{BD}\cdot\cos\beta = \sin\alpha\cos\beta$ and therefore

$$\sin(\alpha+\beta) = CD = CH + HD = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad q.e.d.$$

and for the special case $\alpha = \beta$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$ **.**

To derive the second addition theorem we mirror in Fig. A2.b the triangle BCD at the diameter AB getting the triangle $B\bar{C}\bar{D}$ where $\bar{\beta} = \beta$. We define G according to $x = BG = B\bar{C}$. Since $BG = B\bar{C}$ and $\bar{\beta} = \beta$ the square must form a kite square with $\bar{D}G = C\bar{D} = CD$.



FigA2b: Basic construction to derive $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Lemma of Archimedes: If $arcB\bar{C} = arcBC = 2\beta$ and if $BH \perp AC$ then it must be $H\bar{D} = CH + CD$.
Using $H\bar{D} = \sin\alpha\cos\beta$, $CH = \cos\alpha\sin\beta$ and $CD = (\sin\alpha - \beta)$ one gets
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ *q.e.d.*

Back to Archimedes goes also the following derivation of the goniometric formula $\sin(\alpha/2) = \sqrt{(1-\cos\alpha)/2}$, used by Ptolemy in the Almagest (see Neugebauer 1975, p.23).

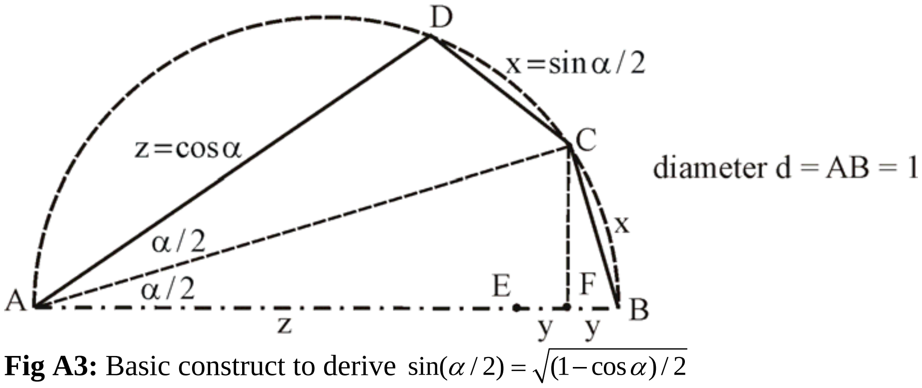


Fig A3: Basic construct to derive $\sin(\alpha/2) = \sqrt{(1-\cos\alpha)/2}$

In fig. A3 the point E is chosen such that AD=AE=z; one gets x=BC=CD=CE=$\sin(\phi/2)$. Since ECB is an isosceles triangle its altitude at F must divide EB into two equal parts y.
For the rectangular triangle ACB one gets first

$2y = 1 - z = 1 - \cos\alpha$ or $y = (1 - \cos\alpha)/2$.

From the two rectangular triangles AFB and ACB one gets the proportion

$\frac{x}{1} = \frac{y}{x}$ or $x^2 = y$ or $x = \sqrt{y}$.

Combining the relations one gets

$\sin(\alpha/2) = x = \sqrt{y} = \sqrt{(1-\cos\alpha)/2}$.

Furthermore, one gets simply

$\cos^2(\alpha/2) = 1 - \sin^2(\alpha/2) = 1 - (1-\cos\alpha)/2 = (1+\cos\alpha)/2$

and therefore

$\cos(\alpha/2) = \sqrt{(1+\cos\alpha)/2}$.

Using $\bar{\alpha} = \alpha/2$ one gets simply $2\sin^2\bar{\alpha} = (1 - \cos 2\bar{\alpha})$ or for $\bar{\alpha} = \alpha$

$\cos 2\alpha = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha$.

Archimedes had obviously also all goniometric relations for double and half angles at his disposal, very important to establish a sine-table.

In order to establish a convenient trigonometric table the trisection of an angle becomes necessary. Archimedes could have done this by introducing $\beta = 2\alpha$ and, starting with the relation

$$\begin{aligned}\sin 3\alpha &= \sin(\alpha + 2\alpha) = \\ &= \sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha \\ &= \sin\alpha(1 - 2\sin^2\alpha) + 2\sin\alpha\cos^2\alpha \\ &= \sin\alpha(1 - 2\sin^2\alpha) + 2\sin\alpha(1 - \sin^2\alpha)\end{aligned}$$

getting

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

or

$\sin\alpha = (1/3)\sin 3\alpha + (4/3)\sin^3\alpha$.

For small angles such as e.g. 1 cubitus=2.5° this equation can be solved by a few iterations, starting with $(\sin\alpha)_0 = (1/3)\sin^3\alpha$.

Of course, regarding the establishment of a sine-table the use of the smaller cubit 1 cubitus=24daktylos=2° was much more appropriate. Was this the reason why in Alexandria at about 175 bc the angular unit 1 degree=1° was introduced for astronomical purposes starting from sin3°?

In any case it would be hard to believe that Archimedes did not inform by correspondence hi colleagues in Alexandria about those results very important regarding the practical problems in geodesy/ geography as well as astronomy.

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BIOGRAPHICAL NOTES

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