A Calculation Program for Geoid Undulations Using Orthogonal Polynomials

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Key Words: Geoid, GPS, Ellipsoidal Height, Orthometric Height

SUMMARY:

In recent years, the use of GPS (Global Positioning System) for precise point positioning in various applications in Turkey as in other countries has caused serious changes. Three dimensional coordinates which are obtained easily, rapidly and precisely by GPS have been used widespread in several applications such as drawing large scale maps and data collection for geographical information system. The latitude and the longitude obtained by GPS are used directly or after they are transformed, but ellipsoidal heights can't be used in applications. In order to transform the ellipsoidal heights to orthometric heights used in applications, the geoid undulations that have adequate accuracy must be known.

In this study, a computer program called TRANSFORMER in DELPHI language was written for the calculation of geoid undulations according to the fiducial sites by orthogonal polynomials. By this program, an appropriate surface was fitted using the undulations for fiducial sites of which appropriateness of x, y coordinates was proved and geoid undulations are known.

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1. INTRODUCTION

The three dimensional information has been obtained by GPS. Height is the third dimension among these dimensions. Today, when we talk about the height of a point, the vertical distance of the point from the sea level is understood. So, the mean sea level must be determined firstly. In our country, the mean sea level was measured at the tide gauge site of Antalya for 22 years, and the country leveling network was coded according to these results (Arbey, 1988; Inal, 1996). In many countries, the spot elevations were determined according to the mean sea level. The mean sea level constitutes a sufficient reference surface for the construction of height unity in a country. If the unity wanted for the heights all over the earth, the geoid must be taken into account as a reference surface. Geoid is the most important reference surface for heights, and it constitutes a leveling reference surface for heights that best fixes at the mean sea level (Torge, 1980).

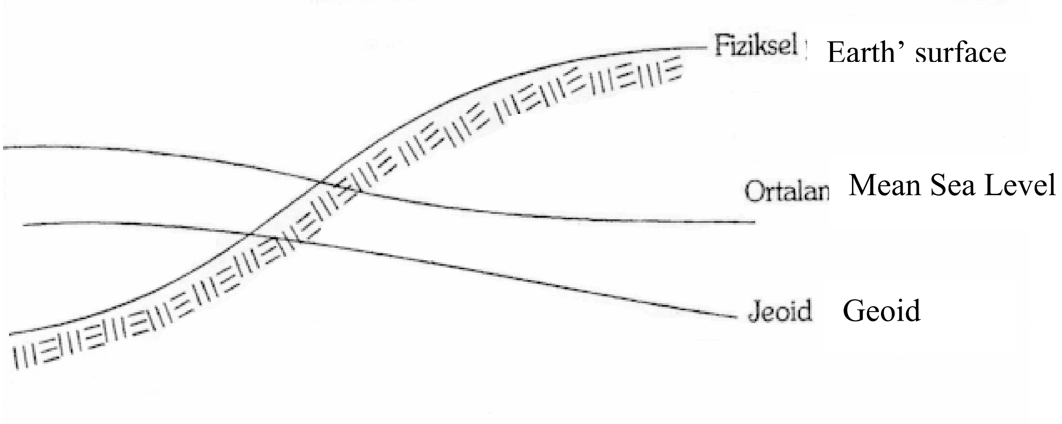

Figure 1. The relation between the mean sea level and geoid

However, today, the height from the geoid and the mean sea level are assumed as the same in many engineering applications. Because of this, we can define the geoid as the best matched potential surface at the mean sea level (İnal, 1996).

2. THE DETERMINATION OF THE GEOID UNDULATIONS BY GPS/LEVELING

The latitude, longitude and ellipsoidal heights are determined in a global geocentric coordinate system for every points on earth by GPS (Global Positioning System). The orthometric heights of points must be used in the engineering works and for drawing maps. The orthometric heights are determined classically by geometric levelling measurements which depend on the National Vertical Control Network. The appropriate geoid models must be determined for transforming the orthometric heights directly from ellipsoidal heights obtained by GPS (Kılıçoğlu, 2002).

The undulation of the geoid as seen in Figure 2 is calculated by the following equation (Liddle, 1989; Ollikainen, 1997).

$$N = h - H \tag{1}$$

where N: the undulation of the geoid, h: ellipsoidal height, H: orthometric height.

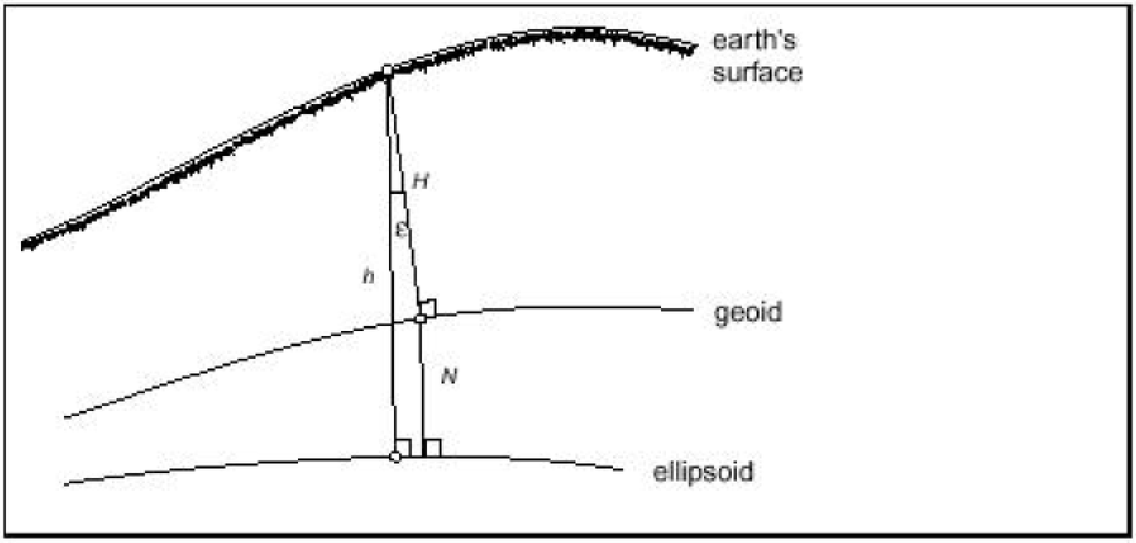

Figure 2 The relation between orthometric and ellipsoidal heights. (Zhan-Ji 1998)

3. THE INTERPOLATION BY POLYNOMIAL SURFACES

One of the most common techniques used for the determination of geoid undulations is to fit an analytical surface which both values of ellipsoidal and orthometric heights are known and to the best shaped points of the geoid are benefited. The mathematical model that is obtained by the surface-fitting is used to find out the value for the middle points of the geoid undulation. The fact must not be forgotten that only the undulation of the geoid values of the middle points can be calculated by the surface-fitting model. The calculated values are used for the orthometric height values. This method is similar to the astrogeodetic method. In both methods; except the errors due to observation, the highest accuracy is obtained from the applications using very close stations in where the area of the geoid is smooth (King et al. 1985).

The interpolation technique with the polynomial surfaces is the most common technique used for surface modeling. The main aim of this technique is to model the working area by one function. The basic feature of this technique is to produce fixed coefficients by using the fiducial points of which values are known to form the model surface and to find out unknown dimensions by using the known dimensions of the new points with the fixed coefficients (İnal and Yiğit, 2004).

For the surface fitting, the following equation can be used:

$$N(x,y)=\sum_{k=0}^{n}\sum_{\substack{j=k-i\\ i=0}}^{k} a_{ij}x^i y^j \tag{2}$$

where,

aij : Unknown coefficient of polynomial,

n : The degree of the surface (1,2,3),

x, y : The plane coordinated of the points,

i, j : (x, y) The positive integer of the bases of the coordinates (Petrie and Kennie,1987).

In equation (2), when the degree of the polynomial is chosen as n=1, it means the surface is linear, when it's chosen as n=2, the surface is quadratic, when it's chosen as n=3 the surface is called as cubic.

If the amount of fiducial sites is more than unknown numbers, the a_{ij} coefficients are calculated by adjustment using the least squares method. A; is the coefficient matrix, L; is the value which shows the undulation of fiducial sites; and then the following equations can be written as those followings:

$$\begin{aligned} N &= A^T A \\ n &= A^T L \\ x &= N^{-1} n \end{aligned} \tag{3}$$

The following equations can be written according to the surface degree in interpolation as polynomials;

$$\begin{aligned} N(x,y) &= a_0 + a_1 y + a_2 x && n=1 \\ N(x,y) &= a_0 + a_1 y + a_2 x + a_3 x^2 + a_4 xy + a_5 y^2 && n=2 \\ N(x,y) &= a_0 + a_1 y + a_2 x + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3 && n=3 \end{aligned} \tag{4}$$

By this method, if the surface degree is 1, at least 3, if the surface degree is 2, at least 6, if the surface degree is 3, at least 10 fiducial sites are needed (İnal, 1996).

4. THE DETERMINATION OF THE OUTLIER POINTS

The outlier points among fiducial sites which are used in the calculation of the transformation parameters must be cleaned. For this, surface function is determined firstly. V_i difference is calculated between the known geoid undulations and geoid undulations obtained using the function below:

$$m_0 = \sqrt{\frac{[VV]}{P}} \tag{5}$$

P shows the number of the fiducial sites used in transformation. The test size is determined by using the equation below:

$$T_i = \frac{|v_i|}{m_0} \tag{6}$$

The determined test sizes are compared with the limit value with the error probability $\alpha = 0.05$ in equation (7).

$$C = \sqrt{(p-1) * (1 - (\frac{\alpha}{p})^{(\frac{1}{p-2})})} \tag{7}$$

The biggest T_i value which is bigger than C limit value is accepted for the determination of outlier point. The operations are repeated after this point was omitted. The same operation is carried out several times until there is no outlier point in the point group (Bektaş and Doğan, 1998).

5. THE INTRODUCTION OF THE DEVELOPED COMPUTER PROGRAM

Programming the transformation in one dimension (the calculation of the geoid undulations by using the orthogonal polynomials), in two dimension and in three dimension coordinate systems are performed by the developed program, transformer. DELPHI programming language was used while developing the program since its visual sides are very strong and the process can be performed as functions, thus transformer can be used in all PC's because of its installation feature.

The control points for the geoid undulations can be calculated by orthogonal polynomials in which the fiducial sites are benefited by transformer program. The program's interface for the one dimensional transformation is seen in figure 3.

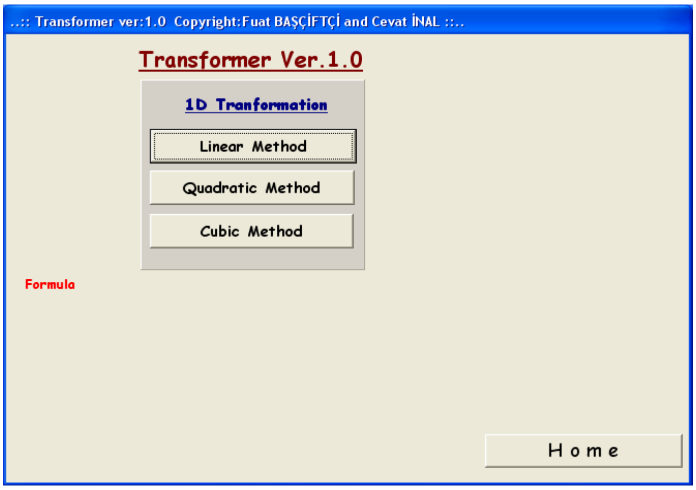

Figure 3. The program’s interface for the one dimensional transformation

The method and the correlation can be seen together after the desired method is chosen (Figure 4).

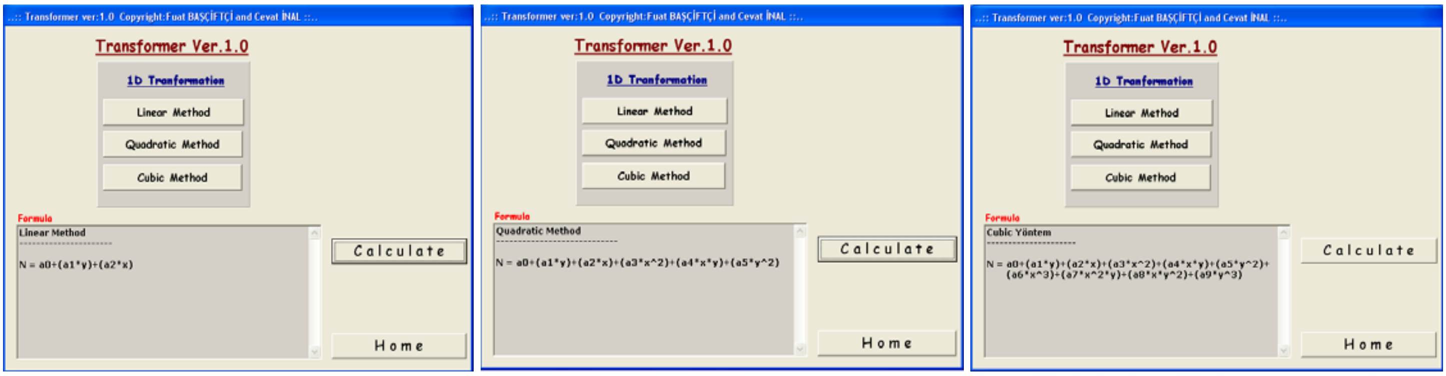

Figure 4. Interfaces for the beginning stages of the program for linear, quadratic, cubic methods.

By clicking on calculate button in these interfaces, the view of the liner method at Figure 5, the quadratic method at Figure 6, the cubic method at Figure 7 can be seen on the screen.

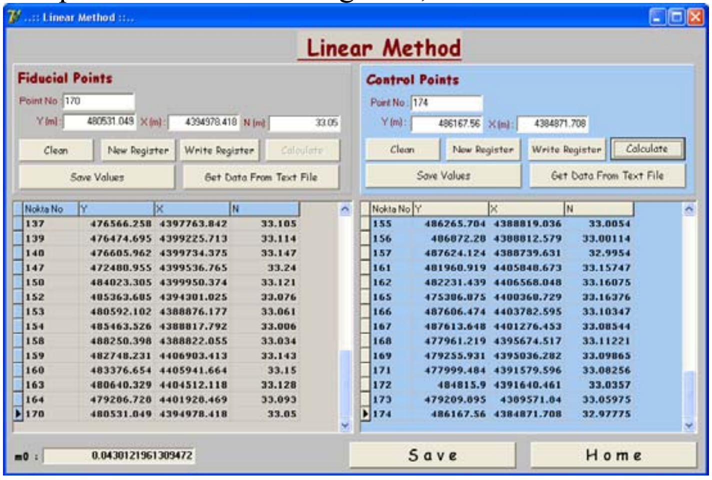

Figure 5. The view of the linear method.

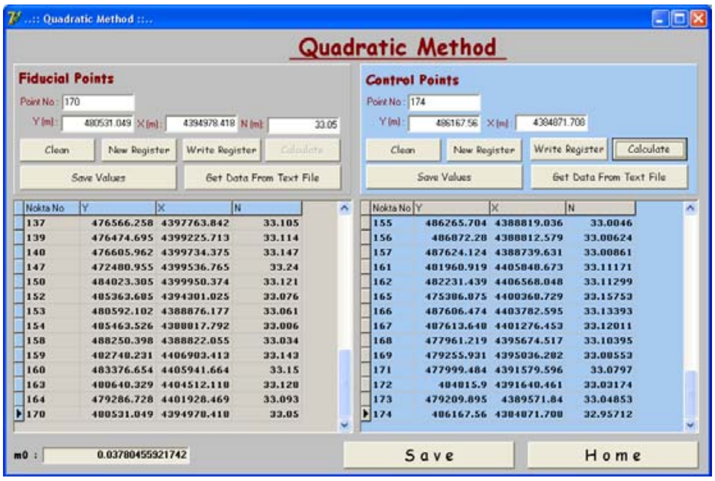

Figure 6. The view of the quadratic method

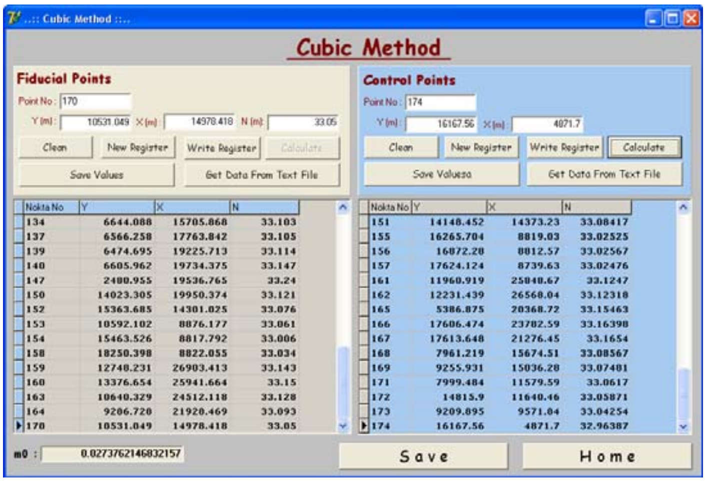

Figure 7. The view of the cubic method

5.1. The File Structure

The point entrances in transformer program can be done either manually, or in the form of a text file in MS Windows. Two text files belong to fiducial sites and control points must be formed. The text file for the fiducial sites includes coordinates of y, x and the geoid undulation (Figure 8), and the file for control points includes y and x coordinates of the points (Figure 9).

File Edit Format View Help

201	457350.771	4203118.107	35.933
205	457866.337	4208316.635	36.062
210	457511.715	4215089.356	36.272
213	456272.562	4220411.955	36.473
215	456171.958	4222220.133	36.529
216	456021.292	4222936.340	36.568
217	450424.493	4215912.329	36.752
219	453093.482	4215451.677	36.545
229	460642.036	4214160.195	36.129
233	466084.501	4214003.820	35.943
234	454491.800	4210558.610	36.169
236	454860.432	4212406.176	36.236
239	456586.978	4214308.555	36.298
244	458297.564	4218800.618	36.423
247	459566.412	4221778.512	36.483
249	458832.911	4223293.606	36.617
250	454939.545	4218862.268	36.499
255	458167.736	4214726.811	36.286
260	461806.250	4210947.468	36.017

Point No　　coordinates (Y, X)　　Jeoid Undulation (N)

Figure 8. The file formed for fiducial sites

File Edit Format View Help

202	457523.397	4204563.944
203	457667.996	4205898.440
204	457744.904	4206634.619
206	457613.344	4209960.022
207	457590.150	4211525.999
208	457593.295	4213115.327
209	457574.876	4214503.764
211	456911.427	4216820.366
212	456691.685	4218980.635
214	456211.485	4221192.704
218	452112.258	4215687.146
220	453868.420	4215363.863
221	455135.332	4215253.747
222	456260.063	4215164.788
223	457276.236	4215413.350
224	458151.340	4215615.592
225	458469.607	4215342.540

Point No

coordinates (Y, X)

Figure 9. The file formed for control points

6. APPLICATION

64 points with Gauss-Kruger coordinates and known geoid undulations were utilized in the work area. Among them, 20 points in an appropriate distribution were taken as the fiducial sites and the other remaining 44 points were chosen as control points (Figure 10). The geoid undulations at the 44 control points were calculated by the transformer program.

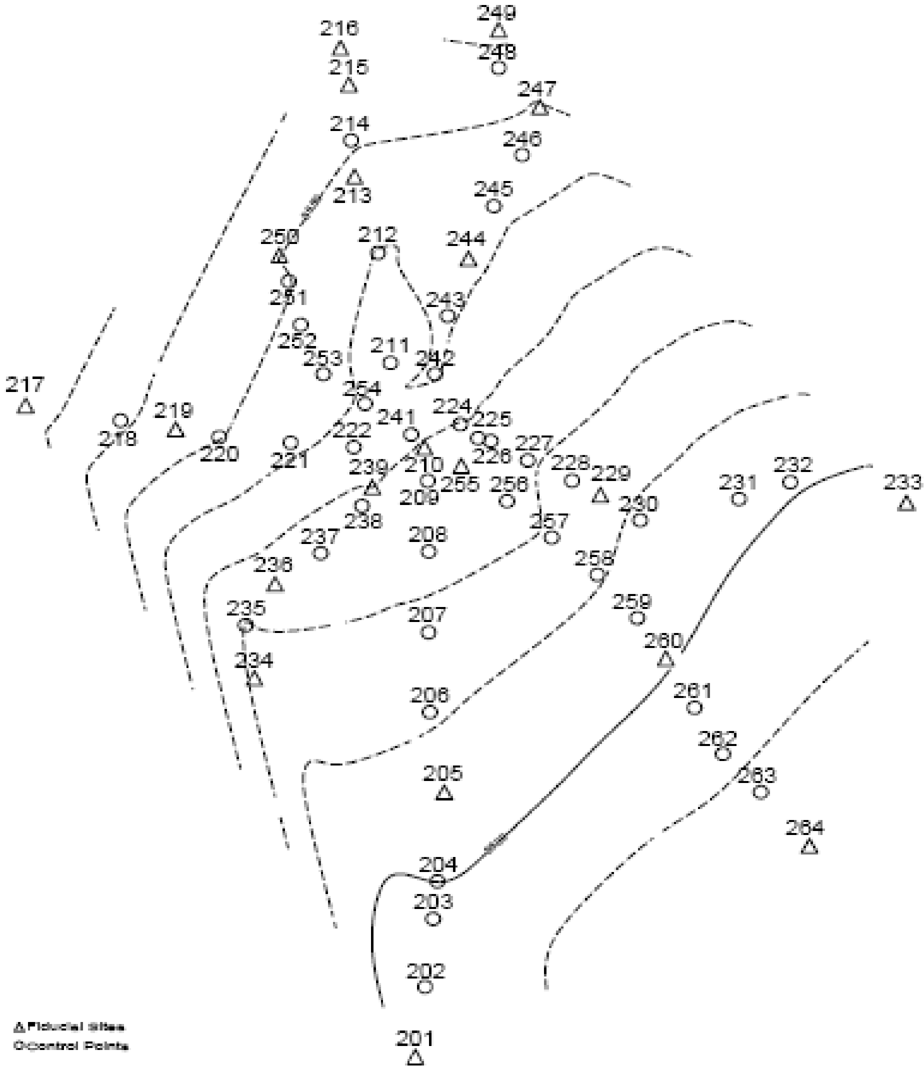

Figure 10. The fiducial sites and the control points

Table 1. The projection coordinates and the geoid heights of the points at the work area.

Fiducial sites

NN	Y	X	N	NN	Y	X	N
201	457350.771	4203118.107	35.933	234	454491.800	4210558.610	36.169
205	457866.337	4208316.635	36.062	236	454860.432	4212406.176	36.236
210	457511.715	4215089.356	36.272	239	456586.978	4214308.555	36.298
213	456272.562	4220411.955	36.473	244	458297.564	4218800.618	36.423
215	456171.958	4222220.133	36.529	247	459566.412	4221778.512	36.483
216	456021.292	4222936.340	36.568	249	458832.911	4223293.606	36.617
217	450424.493	4215912.329	36.752	250	454939.545	4218862.268	36.499
219	453093.482	4215451.677	36.545	255	458167.736	4214726.811	36.286
229	460642.036	4214160.195	36.129	260	461806.250	4210947.468	36.017
233	466084.501	4214003.820	35.943	264	464357.338	4207267.482	35.834

Control points

NN	Y	X	N	NN	Y	X	N
202	457523.397	4204563.944	35.929	232	464016.149	4214476.449	36.007
203	457667.996	4205898.440	35.950	235	454339.130	4211668.936	36.195
204	457744.904	4206634.619	36.011	237	455670.804	4213083.171	36.232
206	457613.344	4209960.022	36.112	238	456403.015	4214012.593	36.265
207	457590.150	4211525.999	36.182	240	456587.758	4214527.809	36.311
208	457593.295	4213115.327	36.228	241	457290.830	4215297.753	36.337
209	457574.876	4214503.764	36.247	242	457701.004	4216603.421	36.413
211	456911.427	4216820.366	36.327	243	457925.832	4217745.558	36.415
212	456691.685	4218980.635	36.394	245	458747.438	4219900.750	36.425
214	456211.485	4221192.704	36.507	246	459252.257	4220905.700	36.456
218	452112.258	4215687.146	36.618	248	458835.202	4222614.580	36.577
220	453868.420	4215363.863	36.505	251	455102.244	4218421.746	36.506
221	455135.332	4215253.747	36.425	252	455313.343	4217570.686	36.482
222	456260.063	4215164.788	36.368	253	455717.110	4216602.242	36.447
223	457276.236	4215413.350	36.328	254	456460.343	4216015.669	36.391
224	458151.340	4215615.592	36.293	256	458981.377	4214105.353	36.235
225	458469.607	4215342.540	36.256	257	459778.632	4213391.261	36.194
226	458693.351	4215286.071	36.251	258	460588.212	4212658.454	36.111
227	459352.483	4214903.733	36.208	259	461299.859	4211809.826	36.058
228	460137.192	4214505.427	36.158	261	462318.250	4210045.476	35.961
230	461348.522	4213729.002	36.084	262	462819.997	4209137.378	35.923
231	463103.007	4214147.253	36.022	263	463491.690	4208389.078	35.876

The outlier measurement test was applied to all orthogonal polynomial surfaces used. According to the test result at the fiducial sites in liner method, the point of 217 was determined as outlier fiducial and the rest of 19 fiducial points and unknown surfaces were re-analyzed. The 20 fiducial sites were found matched to the reference surface according to the quadratic and cubic method. For the 44 control points that were determined by the help of unknowns, the geoid undulations found by GPS/Leveling (Figure11), and the undulation differences are shown in table 2.

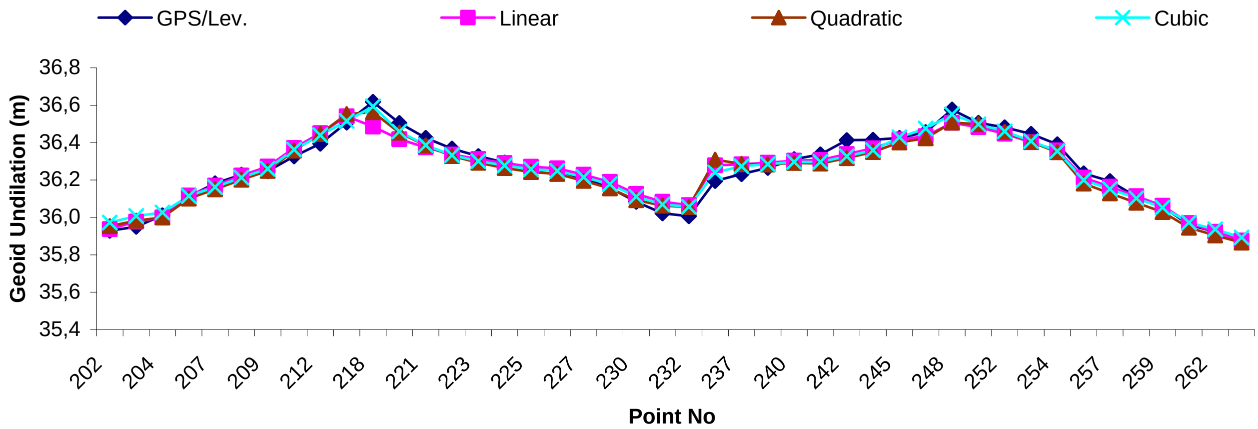

Figure 11. The geoid undulations at the control points.

Table 2. The geoid undulations and the undulation differences at the control point.

Point No	**GPS/Lev. (m)**	**Linear (m)**	**Quadratic (m)**	**Cubic (m)**	**GPS/Lev. Linear (cm)**	**GPS/Lev. Quadratic (cm)**	**GPS/Lev. Cubic (cm)**
202	35.929	35.937	35.954	35.972	-0.84	-2.50	-4.32
203	35.950	35.978	35.983	36.008	-2.79	-3.29	-5.79
204	36.011	36.000	36.000	36.027	1.07	1.14	-1.55
206	36.112	36.117	36.101	36.114	-0.48	1.08	-0.22
207	36.182	36.171	36.151	36.162	1.15	3.13	1.96
208	36.228	36.224	36.202	36.213	0.39	2.62	1.48
209	36.247	36.272	36.249	36.260	-2.46	-0.21	-1.28
211	36.327	36.371	36.358	36.363	-4.40	-3.10	-3.55
212	36.394	36.451	36.447	36.440	-5.70	-5.28	-4.56
214	36.507	36.541	36.553	36.516	-3.40	-4.59	-0.90
218	36.618	36.486	36.563	36.594	13.24	5.47	2.40
220	36.505	36.419	36.453	36.459	8.63	5.17	4.60
221	36.425	36.375	36.385	36.387	5.04	4.04	3.84
222	36.368	36.336	36.328	36.333	3.22	3.96	3.54
223	36.328	36.312	36.293	36.301	1.62	3.52	2.68
224	36.293	36.291	36.265	36.278	0.22	2.83	1.53
225	36.256	36.271	36.243	36.258	-1.55	1.30	-0.20
226	36.251	36.262	36.233	36.249	-1.14	1.83	0.20
227	36.208	36.229	36.196	36.216	-2.05	1.23	-0.76
228	36.158	36.190	36.156	36.178	-3.21	0.24	-1.98
230	36.084	36.125	36.092	36.111	-4.12	-0.83	-2.69
231	36.022	36.084	36.063	36.069	-6.15	-4.08	-4.73
232	36.007	36.066	36.056	36.057	-5.86	-4.89	-5.03
235	36.195	36.279	36.309	36.240	-8.39	-11.35	-4.50
237	36.232	36.284	36.286	36.274	-5.22	-5.44	-4.20
238	36.265	36.292	36.283	36.285	-2.73	-1.79	-1.99
240	36.311	36.304	36.292	36.297	0.72	1.90	1.44
241	36.337	36.307	36.288	36.297	2.95	4.88	4.02
242	36.413	36.339	36.318	36.327	7.45	9.54	8.64
243	36.415	36.370	36.350	36.359	4.50	6.47	5.56
245	36.425	36.417	36.402	36.428	0.84	2.35	-0.32
246	36.456	36.434	36.424	36.476	2.15	3.23	-2.00
248	36.577	36.506	36.507	36.551	7.15	6.97	2.56
251	36.506	36.483	36.499	36.498	2.33	0.75	0.78
252	36.482	36.447	36.457	36.461	3.48	2.47	2.14
253	36.447	36.402	36.404	36.408	4.53	4.34	3.92
254	36.391	36.358	36.349	36.354	3.28	4.20	3.71
256	36.235	36.213	36.181	36.200	2.17	5.37	3.49
257	36.194	36.164	36.129	36.152	3.02	6.47	4.22
258	36.111	36.113	36.079	36.102	-0.23	3.25	0.86
259	36.058	36.062	36.029	36.053	-0.40	2.88	0.53
261	35.961	35.970	35.945	35.972	-0.89	1.63	-1.13
262	35.923	35.923	35.904	35.936	-0.03	1.92	-1.34
263	35.876	35.877	35.865	35.893	-0.06	1.07	-1.72

In order to determine the one that gives the best result among the applied orthogonal polynomial surfaces, the root mean square errors of the unit measurement calculated by the equation (5) which were determined by the deviations from the fiducial sites of the surface point. The root mean square errors are shown at table 3.

Table 3. The root mean square errors determined from the interpolation of the orthogonal polynomials.

	Liner **m_0 (cm)**	**Quadratic** **m_0 (cm)**	**Cubic** **m_0 (cm)**
Step 1	±6.36	±6.11	±3.13
Step 2	±4.86		

7. CONCLUSION

The degree of the surface polynomial that were used for the working area cannot be estimated previously. To determine this, the degree of the surface polynomial can be determined by statistical analysis of the adjusted results starting from the first degree polynomial. The last variance value gets smaller when the polynomial degree increases. The one lower degree than the polynomial degree of the latest variance when it is started to increase is accepted as the most appropriate degree.

In this study, a program called transformer was developed in the Delphi to compute the geoid undulations by orthogonal polynomials. The x, y coordinates of the fiducial sites proved in matching and transformation parameters using geoid undulations can be determined by this program. The geoid undulations of the control points at the test area were calculated by the developed program, and the calculated values and the known points of GPS/Leveling values were compared.

The results of the comparison can be summarized as followings;

- The difference between GPS/Leveling and the geoid undulations determined by the linear method changes between 8.39 and 13.24 cm, and the root mean square error of this case is (m_0) ±4.86 cm,
- The difference between GPS/Leveling and the geoid undulations determined by the quadratic method changes between -11.35 and 9.54 cm and the root mean square error for this case is (m_0) ±6.11 cm,
- The difference between GPS/Leveling and the geoid undulations determined by the cubic method changes between -5.79 cm and 8.64 cm and the root mean square error for this case is (m_0) ±3.13 cm,
- The cubic method shows a good approache to the determined values by GPS/Leveling at the work area.

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BIOGRAPHICAL NOTES

Fuat Basciftci, born in 1978. I was graduated from Selcuk University, Engineering and Architecture Faculty, Geodesy and Photogrammetry Engineering Department in 2001. I started working as a lecturer at Selcuk University, Kadinhani Faik İçil Vocational School of Higher Education, Mapping-Cadastre Programme in the same year. I graduated from the Master Education in 2008. My interests are geoid determination, transformations and GPS.

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